Fractional vortex lattice structures in spin triplet superconductors

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Motivated by recent interest in spin-triplet superconductors, we investigate the vortex lattice structures for this class of unconventional superconductors. We discuss how the order parameter symmetry can give rise to U(1)×U(1) symmetry in the same sense as in spinor condensates, making half-quantum vortices (HQVs) topologically stable.We then calculate the vortex lattice structure of HQVs, with particular attention on the roles of the crystalline lattice, the Zeeman coupling and Meissner screening, all absent in spinor condensates. Finally, we consider how spin.orbit coupling leads to a breakdown of the U(1)×U(1) symmetry in free energy and whether the HQV lattice survives this symmetry breaking. As examples, we examine simpler spin-triplet models proposed in the context of $Na_xCoO_2 \cdot yH_2O$ and Bechgaard salts, as well as the better known and more complex model for Sr_2RuO_4.

I. INTRODUCTION

A half quantum vortices (HQV) with vorticity $h/4e$, which is half that of usual Abrikosov vortex with vorticity $\Phi_0 \equiv h/2e$, presents an exciting example of fractionalized topological defects. Quantization of collective topological defects provides clear cut access to the nature of the ground state. For instance, the vorticity $h/2e$ of Abrikosov vortex in a type II superconductor clearly shows that the circulation associated with the vortex is that of charge $2e$ Cooper pairs. Since the vorticity in a condensate is determined by the requirement of single-valuedness of the order parameter describing the condensate, "fractionalization of vorticity" is possible with a multi-component order parameter when different components are allowed to wind separately. For instance, in a triplet superconductor, the additional Cooper pair spin degree of freedom can be free to rotate in plane[1,2,3] giving rise to additional U(1) symmetry and an associated spin winding number; cases where vortex fractionalization is due to the U(1)×U(1) symmetry of different physical origin has also been studied[4]. Hence observation of fractionalization of vortices can serve as an indicator of the structure of the order parameter in a given condensate. In addition, the recent proposals predicting non-Abelian fractional statistics for the composite of a HQV and the Majorana fermions bound at its core in the chiral triplet superconductors brought in recent rise in the attention and interest to the possibility of HQV's in triplet superconductors[5,6,7,8]. This type of non-Abelian statistics was first studied for quasiholes in the spin-polarized $\nu = 5/2$ quantum Hall state[9,10]. Therefore, if we want to obtain the same statistics for vortices in a spinful superconductor, the vortices should be HQVs so that there would be phase winding only for a single component.

However, while there are a number of candidate triplet superconductors such as the single layer ruthenate Sr_2RuO4[11,12], the cobaltate $Na_xCoO_2 \cdot yH_2O$[13] and organic superconductors[14], HQV's have never been observed in bulk systems, in line with the energetic stability issues raised by two of us in Ref. 3. It was pointed out in Ref. 3 due to the absence of screening for the spin supercurrent circulation required for HQV in triplet superconductors, HQV's can be energetically unstable in bulk samples towards combining into full Abrikosov vortices despite their advantage in magnetic energy. Related considerations have appeared in the context of spin-triplet superconductivity in UPt_3 by Zhitomirsky[15].

The main motivation of this work is to investigate the possibility of using high enough fields to generate a HQV lattice in triplet superconductors where the vortex lattice serves two purposes at once: 1) stabilizing HQV's at finite separation 2) providing an unambiguous signature of its formation (halving of the vortex lattice unit cell). Experiments have already determined the vortex lattice structure successfully at low fields in Sr_2RuO_4[16,17] and the observed square lattice geometry was consistent with the theoretical prediction by one of the present authors based on a chiral triplet order parameter in Ref. 18. However, Ref. 18 considered the limit of strong spin-orbit coupling, which leads to vortex lattices of full quantum Abrikosov vortices. Recently, measurements of the Knight shift for the field along the c-axis[19] as well as ARPES data[20,21] indicate that that spin-orbit coupling is perhaps not so strong. Therefore, in this paper, we extend the studies of Ref. 18 to allow for weak spin-orbit coupling, leading to the possibility of HQV lattices for fields along the c-axis. Additionally, the organic superconductor $(TMTSF)_2ClO_4$ naturally has weak spin-orbit coupling and Knight shift measurements provide evidence for a spin-triplet state at high magnetic fields[14], which is precisely the situation we consider here. We also provide an analysis of this case and the closely related case for cobaltate spin-triplet superconductors[13].

In this paper, we study the energetics of different vortex lattice configurations. The key additional physical ingredient is the U(1) spin-rotational invariance of the Cooper pairs that arises in a magnetic field. This generically leads to two different species of fractional vortices whose fractional fluxes sum to Φ_0. When stable, these fractional vortices form interlacing lattices analogous to vortex.antivortex lattice configurations proposed by Refs. Gabay and Kapitulnik[22], Zhang[23] in the context of a two-dimensional (2D) superfluid and the configuration in two-component Bose condensates proposed by Ref. Mueller and Ho[24].

The rest of the paper consists of the following. In section II, we give a pedagogical introduction to the symmetry properties of triplet OP. In particular, we will show how U(1)×U(1) symmetry can arise in the OP of such systems. In section III we discuss the form of Gibbs free energy that is allowed by various symmetries in the problem when spin-orbit coupling is not included. In section IV we provide the general theoretical framework for the vortex lattice phases. In section V we show that the lowest Landau level solution often provides an adequate description and we discuss this solution for the lattice of HQV's. In section VI we consider the effect of U(1)×U(1) symmetry breaking driven by spin-orbit coupling. In section VII we lay out predictions for how to detect the proposed HQV lattice structures and we conclude with a summary and outlook in section VIII.

II. THE TRIPLET ORDER PARAMETER

The order parameter of a triplet superconductor takes a matrix form in the spin space[12,25]:

$$\hat{\Delta}(\mathbf{k}) = \begin{bmatrix} \Delta_{\uparrow\uparrow}(\mathbf{k}) & \Delta_{\uparrow\downarrow}(\mathbf{k}) \\ \Delta_{\downarrow\uparrow}(\mathbf{k}) & \Delta_{\downarrow\downarrow}(\mathbf{k}) \end{bmatrix} \equiv \begin{bmatrix} -d_x + id_y & d_z \\ d_z & d_x + id_y \end{bmatrix}, \tag{2.1}$$

where the spin quantization axis is along the z direction. The triplet pairing requires $\Delta_{\uparrow\downarrow} = \Delta_{\downarrow\uparrow}$ and a set of three complex functions of $\mathbf{k}$, namely $(d_x(\mathbf{k}), d_y(\mathbf{k}), d_z(\mathbf{k}))$, were introduced to parameterize the gap matrix. When the three functions are collectively represented using a vector notation, the "unit vector" $\hat{\mathbf{d}}(\mathbf{k})$ represents the symmetry direction (zero projection direction) with respect to the rotation of Cooper pair spin. In the presence of the sufficiently high field along the c-axis, Zeeman splitting between electrons with opposite spins prohibits pairing, leading to $\Delta_{\uparrow\downarrow} = \Delta_{\downarrow\uparrow} = 0$. In the $\mathbf{d}$-vector notation, this implies that the $\mathbf{d}$-vector lies in-plane (perpendicular to the applied field). In the rest of the paper, we assume that the field is sufficiently large so that this is the case. In the context of strontium ruthenate, our results apply for the field along the c-axis (this is also true for the cobaltates when we are discussing spin-orbit coupling in hexagonal systems). For organic and cobaltate superconductors, our results apply for the field along any two-fold or higher symmetry axis of symmetry.

For non-chiral triplet order parameter symmetry, which is expected of the cobaltate $Na_xCoO_2 \cdot yH_2O$[13] and organic superconductors[14], the spin pairing gap matrix takes the form

$$\hat{\Delta}(\mathbf{k}) = f(\mathbf{k}) \begin{bmatrix} \Delta_{\uparrow\uparrow} & 0 \\ 0 & \Delta_{\downarrow\downarrow} \end{bmatrix} \tag{2.2}$$

where the function $f(\mathbf{k})$ depends on the specific odd angular momentum channel. The key simplifying feature is the the orbital dependence is described by a one-dimensional representation encoded by $f(\mathbf{k})$. There has been suggestions that the cobaltate $Na_xCoO_2 \cdot yH_2O$ has a spin-triplet pairing through f-wave channel[13], although data from the Knight shift experiments remain controversial[26,27]. In this case a common choice is $f(\mathbf{k}) = k_x(k_x^2 - 3k_y^2)$. However, the precise form of $f(\mathbf{k})$ is not needed for our results. For the organic superconductor $(TMTSF)_2ClO_4$, there is a strong case that the system becomes a triplet superconductor under sufficient field $H \gtrsim 20$kOe. In this case, there are many proposals for $f(\mathbf{k})$. However, again, the specific form is not needed for our results.

For chiral order parameter symmetry expected of the Sr_2RuO_4, with $\mathbf{d}$ in the basal plane and no spin-orbit coupling, the order parameter has four complex degrees of freedom:

$$\hat{\Delta}(\mathbf{k}) = \sum_{\sigma=\pm} \tilde{f}(k_\sigma, k_z) \begin{bmatrix} \Delta_{\uparrow\uparrow,\sigma} & 0 \\ 0 & \Delta_{\downarrow\downarrow,\sigma} \end{bmatrix}, \tag{2.3}$$

where the function $\tilde{f}$, like the non-chiral case discussed above, depends on the specific odd angular momentum channel and $k_\sigma = k_x + i\sigma k_y$. This is equivalent to

$$\mathbf{d}(\mathbf{k}) = \Delta_+ \hat{\mathbf{d}}_+ \exp(in\varphi_{\hat{\mathbf{k}}}) + \Delta_- \hat{\mathbf{d}}_- \exp(-in\varphi_{\hat{\mathbf{k}}}), \tag{2.4}$$

where n is an integer ($n = 1$ for p-wave, $n = 3$ for f-wave) and $\varphi_{\hat{\mathbf{k}}}$ is the azimuthal angle associated with a unit vector $\hat{\mathbf{k}}$ in the 2D plane (assuming quasi-2D setting with the angular momentum along the c-axis: $\hat{\mathbf{l}} = \hat{\mathbf{z}}$). Although $\Delta_- = 0$ for homogeneous chiral superconductor, we will show that $\Delta_- \neq 0$ often plays an important role in describing the vortex lattice structure of chiral superconductor.

Eqs.(2.2) and (2.4) clearly shows that under these circumstances, the order parameter symmetry takes the $U(1) \times U(1)$ form, which can allow for HQV's with $h/4e$ vorticity associated with π orbital phase winding and π $\mathbf{d}$-vector winding.

III. THE GIBBS FREE ENERGY

In order to identify stable vortex type and the lattice structure itself, we start with the Gibbs free energy including all the terms allowed by symmetry up to quartic order. As usual, the quartic terms determine the vortex lattice structure. Due to additional spin degrees of freedom, the full expression for the Gibbs free energy involves a number of additional terms compared to singlet superconductor case and it is instructive to consider different contributions separately:

$$f = f_{mag} + f_0^{(2)} + f_Z^{(2)} + f_{in}^{(2)} + f_{SO}^{(2)} + f_{hom}^{(4)} + f_{in}^{(4)}. \tag{3.1}$$

where $f_{mag} = h^2/8\pi - hH/4\pi$ is the magnetic energy (the field h is the sum of the external field H and the screening field), the superscript (2) indicates terms quadratic in OP and (4) quartic in OP

Other than the conventional quadratic term $f_0^{(2)}$:

$$f_0^{(2)} = -\alpha \sum_i |\Delta_i|^2 \tag{3.2}$$

the remaining quadratic terms in Eq.(3.1) are consequences of additional spin degree of freedom for the triplet superconductors. The Zeeman coupling term

$$f_Z^{(2)} = -\tilde{\kappa} h(|\Delta_{\uparrow\uparrow}|^2 - |\Delta_{\downarrow\downarrow}|^2), \tag{3.3}$$

plays an important role for the HQV vortex lattice by introducing a slight spin-polarization. This slight spin-polarization gives rise to two phase transitions as in the case of the A_1/A_2 phase of ^3He[28,29]. The inhomogeneous part of the quadratic terms $f_{in}^{(2)}$ are of the form

$$K_{ij;kl}(D_i\Delta_k)(D_j\Delta_l)^* + \text{c.c.}, \tag{3.4}$$

where $D_i = \nabla_i + (2\pi i/\Phi_0)A_i$. For these terms, we require rotational invariance up to the lattice symmetry with respect to orbital degrees of freedom only, which means that we require invariance with respect to rotating D_i's and the orbital component of Δ_i's. The complex structure of $f_{in}^{(2)}$ can result in the condensate wave function of a different form than that of conventional SC. $f_{SO}^{(2)}$ is the quadratic spin-orbit coupling term assuming the spin-orbit coupling to be small and is discussed in Sec. VI. In the presence of spin-orbit coupling the free energy have to be invariant under the combined discrete rotation of the orbital and spin degree of freedom specific for the given lattice symmetry. For lattices with orthogonal or tetragonal symmetry, spin-orbit coupling may reduce the symmetry of the Gibbs free energy and tend to suppress HQV formation by introducing a length scale beyond which the HQV's cannot exist (this length scale diverges as the spin-orbit coupling vanishes). This implies that the vortex lattice spacing must be less than this length scale for the HQV lattice to appear. However, we show that for spin-triplet hexagonal materials (specifically the two-dimensional Γ_6^- and Γ_5^- representations in the notation of Sigrist and Ueda[25]), even large spin-orbit coupling still allows for the existence of a fractional vortex lattice. This consideration may apply to $Na_xCoO_2{\cdot}yH_2O$.

Among the quartic terms, $f_{hom}^{(4)}$ represents the usual set of homogeneous terms. As is shown in Appendix A, certain quartic terms vanish in the weak-coupling theory. The terms that vanish are those that lift the energy degeneracy between the full quantum vortex (QV) and the HQV lattice. For this reason, we also include the inhomogeneous quartic term $f_{in}^{(4)}$. This term accounts for the difference between the spin phase stiffness ρ_{sp} and the overall phase stiffness ρ_s. Not only does this difference play an important role in the stability of isolated HQV's as it was shown in Ref. 3 it plays the role of tuning parameter for the vortex lattice structure.

With multiple systems in mind, we consider contributions to the Gibbs free energy specific for non-chiral and chiral superconductors respectively.

A. Non-chiral Triplet Superconductor

As mentioned earlier, here we assume the orbital dependence of the gap function to be the same for all spin-triplet components. Formally, this means that the orbital degree of freedom belongs to a one-dimensional irreducible representation of the point group. We apply our analysis to materials that have orthorhombic, tetragonal, or hexagonal point groups. One relevant example is a non-chiral triplet f-wave superconductor with hexagonal symmetry, which has been proposed in the context of the the cobaltates $Na_xCoO_2 \cdot yH_2O$; in this case $f(\mathbf{k}) = k_x(k_x^2 - 3k_y^2)$ from Eq.(2.2). When the **d**-vector lies in the xy plane, the inclusion of spin-orbit coupling implies that formally this order parameter belongs to the Γ_6^- representation of the hexagonal point group (the consequences of spin-orbit coupling for this representation is discussed in more detail in Section VI).

With the in-plane spin rotational invariance, the relevant free energy within the assumptions stated above is given by

$$f_{in}^{(2)} = \sum_{i=x,y,z} K_i(|D_i\Delta_{\uparrow\uparrow}|^2 + |D_i\Delta_{\downarrow\downarrow}|^2), \tag{3.5}$$

$$f_{hom}^{(4)} = \beta_1(\sum_i |\Delta_i|^2)^2 + \beta_2|\Delta_{\uparrow\uparrow}|^2|\Delta_{\downarrow\downarrow}|^2, \tag{3.6}$$

$$f_{in}^{(4)} = \gamma[\Delta_{\uparrow\uparrow}^*\Delta_{\downarrow\downarrow}(\mathbf{D}_\perp\Delta_{\uparrow\uparrow})\cdot(\mathbf{D}_\perp\Delta_{\downarrow\downarrow})^* + \text{c.c}], \tag{3.7}$$

For tetragonal and hexagonal point groups $K_x = K_y$ while for orthorhombic point groups, $K_x \neq K_y$. For the high field limit we are considering, it is possible to re-scale lengths in two directions perpendicular to applied field such that $\tilde{K}_i = \tilde{K}_j$ for $i \neq j$ (where $\tilde{K}_i$ refers to the new coefficient in the re-scaled coordinates). We will therefore ignore the difference between the K_i and assume that for orthorhombic point groups we are working in re-scaled co-ordinates. The term $f_{in}^{(4)}$ is not the most general such term allowed by symmetry. However, it is this term that allows the GL theory to give the same physics as in Ref. 3. Indeed, one can gain more insight into the vortex lattice solutions that minimizes Eq.(3.2) and Eqs.(3.6-3.7) by relating the coefficient of the inhomogeneous quartic term γ to the stiffness ratio $\rho_{sp} < \rho_s$ which controls the energetic stability of a pair of HQV's[3]. Within the London approximation, the gradient terms in Eq.(3.5) and Eq.(3.7) amounts to phase bending energy which will be proportional to $(\rho_s + \rho_{sp})$ and $(\rho_s - \rho_{sp})$ respectively for $\Delta_{\uparrow\uparrow}$ and $|\Delta_{\downarrow\downarrow}|$. Combining Eq.(3.5) and Eq.(3.7) with the homogeneous solution

$|\Delta_{\uparrow\uparrow}|^2 = |\Delta_{\downarrow\downarrow}|^2 = \alpha/(\beta_1 - \beta_2)$, we obtain the following relation between γ and ρ_{sp}/ρ_s:

$$\gamma = \frac{K_1(\beta_1 - \beta_2)}{\alpha} \frac{1 - \rho_{sp}/\rho_s}{1 + \rho_{sp}/\rho_s}. \tag{3.8}$$

Hence $\gamma > 0$ would imply stability of HQV's and double transitions into two possible vortex phases: a lattice of ordinary Abrikosov vortices and a lattice of HQV's. This transition is determined by the β_2 term of Eq. (3.6) and the γ term of Eq. (3.7).

B. Chiral Triplet Superconductor

With the ruthenate Sr_2RuO_4 in mind, we consider a chiral triplet p-wave superconductor with square symmetry for which $\tilde{f}(k_\sigma) = k_x + i\sigma k_y$ in the Eq.(2.3) with

$$\hat{\Delta}(\mathbf{k}) = \sum_{\sigma=\pm} (k_x + i\sigma k_y) \begin{bmatrix} \Delta_{\uparrow\uparrow,\sigma} & 0 \\ 0 & \Delta_{\downarrow\downarrow,\sigma} \end{bmatrix}, \tag{3.9}$$

where $\Delta_{s,\sigma}$ ($s = \uparrow\uparrow, \downarrow\downarrow$ and $\sigma = \pm$) form expansion parameters for the Gibbs free energy. In terms of the $\mathbf{d}$-vector notation $\mathbf{d} \equiv \hat{\mathbf{x}}(\eta_{xx}k_x + \eta_{xy}k_y) + \hat{\mathbf{y}}(\eta_{yx}k_x + \eta_{yy}k_y)$,

$$\begin{aligned} \Delta_{\uparrow\uparrow,+} &= -\ (\eta_{xx} - i\eta_{xy} - i\eta_{yx} - \eta_{yy})/2, \\ \Delta_{\uparrow\uparrow,-} &= -\ (\eta_{xx} + i\eta_{xy} - i\eta_{yx} + \eta_{yy})/2, \\ \Delta_{\downarrow\downarrow,+} &= \quad (\eta_{xx} - i\eta_{xy} + i\eta_{yx} + \eta_{yy})/2, \\ \Delta_{\downarrow\downarrow,-} &= \quad (\eta_{xx} + i\eta_{xy} + i\eta_{yx} - \eta_{yy})/2. \end{aligned} \tag{3.10}$$

Formally, without spin-orbit coupling, this order parameter is a direct product of a E_u orbital representation of the tetragonal point group and the in-plane vector representation for spin rotations. When spin-orbit is included the order parameter contains the four different one-dimensional representations of the tetragonal point group. In the case without spin-orbit coupling, the relevant free energy for this representation can be constructed using the known free energy for the E_u representation[25], we list below $f_{in}^{(2)}$, $f_{hom}^{(4)}$ and $f_{in}^{(4)}$. Before listing $f_{in}^{(2)}$, we note that this free energy term should respect the C_4 symmetry on the xy plane only for the orbital degrees of freedom:

$$(D_x, D_y, \Delta_{s,+}, \Delta_{s,-}) \to (D_y, -D_x, i\Delta_{s,+}, -i\Delta_{s,-}). \tag{3.11}$$

However, for simplicity, we consider cylindrical symmetry in the orbital degrees of freedom, this does not significantly alter the arguments below. This symmetry gives us $f_{in}^{(2)} = \sum_s f_{in}^{(2,s)}$ where

$$\begin{aligned} f_{in}^{(2,s)} &= K_1(|\mathbf{D}\Delta_{s,+}|^2 + |\mathbf{D}\Delta_{s,-}|^2) \\ &+ K_2[\{(D_x\Delta_{s,+})(D_x\Delta_{s,-})^* - (D_y\Delta_{s,+})(D_y\Delta_{s,-})^*\}/2 \\ &+ \{(D_x\Delta_{s,-})(D_x\Delta_{s,+})^* - (D_y\Delta_{s,-})(D_y\Delta_{s,+})^*\}/2 \\ &+ i\{(D_x\Delta_{s,-})(D_y\Delta_{s,+})^* + (D_y\Delta_{s,-})(D_x\Delta_{s,+})^*\}/2 \\ &- i\{(D_x\Delta_{s,+})(D_y\Delta_{s,-})^* + (D_y\Delta_{s,+})(D_x\Delta_{s,-})^*\}/2] \\ &+ K_4(|D_z\Delta_{s,+}|^2 + |D_z\Delta_{s,-}|^2). \end{aligned} \tag{3.12}$$

In addition, the following term is also allowed by symmetry

$$\delta K \frac{2\pi}{\Phi_0} h \sum_s (-|\Delta_{s+}|^2 + |\Delta_{s-}|^2), \tag{3.13}$$

and it stabilizes this in-plane chiral phase for strong enough magnetic field (note the similarity to the Zeeman term for the condensate spin degrees of freedom). As for the homogeneous quartic terms,

$$\begin{aligned} f_{hom}^{(4)} &= \sum_s [\beta_1(|\Delta_{s+}|^4 + |\Delta_{s-}|^4)/2 + \beta_1'|\Delta_{s+}|^2|\Delta_{s-}|^2] \\ &- \sum_{\sigma=\pm} (\beta_2|\Delta_{\uparrow\uparrow,\sigma}|^2|\Delta_{\downarrow\downarrow,\sigma}|^2 + \beta_2'|\Delta_{\uparrow\uparrow,\sigma}|^2|\Delta_{\downarrow\downarrow,-\sigma}|^2) \\ &- \beta_3[(\Delta_{\uparrow\uparrow,+}\Delta_{\downarrow\downarrow,-})(\Delta_{\uparrow\uparrow,-}\Delta_{\downarrow\downarrow,+})^* + \text{c.c.}]. \end{aligned} \tag{3.14}$$

β_2, β_2' and β_3 terms originate from interaction between spin up-up pairs and down-down pairs. Again, for simplicity, we written the free energy in the limit of a cylindrical Fermi surface. Lastly, we have

$$\begin{aligned} f_{in}^{(4)} &= \gamma \sum_{\sigma=\pm} [\Delta^*_{\uparrow\uparrow,\sigma}\Delta_{\downarrow\downarrow,\sigma}(\boldsymbol{D}\Delta_{\uparrow\uparrow,\sigma}) \cdot (\boldsymbol{D}\Delta^*_{\downarrow\downarrow,\sigma}) + \text{c.c.}] \\ &+ \gamma' \sum_{\sigma=\pm} [\Delta^*_{\uparrow\uparrow,\sigma}\Delta_{\downarrow\downarrow,-\sigma}(\boldsymbol{D}\Delta_{\uparrow\uparrow,\sigma}) \cdot (\boldsymbol{D}\Delta^*_{\downarrow\downarrow,-\sigma}) + \text{c.c.}]. \end{aligned} \tag{3.15}$$

Note that the form of Eq. (3.15) is consistent with the form of the interaction terms in Eq. (3.14). Again, this is not the most general term allowed by symmetry, but it is the minimal term that captures the physics in the London limit described in Ref. 3.

IV. DETERMINING THE VORTEX LATTICE STRUCTURE - GENERAL CONSIDERATIONS

We consider the vortex lattice phases near the upper critical field to map out the stability condition for HQV lattice phases. As usual, the first step towards determining the vortex lattice structure is to identify the eigenstates of the linearized Ginzburg-Landau (GL) equations. In order to obtain a linearized GL equation we take a variation of the quadratic terms in the free energy, for example:

$$\begin{aligned} f_0^{(2)} + f_{in}^{(2)} + f_Z^{(2)} &= -\alpha \sum_j |\Delta_j|^2 - \tilde{\kappa}h(|\Delta_{\uparrow\uparrow}|^2 - |\Delta_{\downarrow\downarrow}|^2) \\ &+ [K_{jk;lm}(D_j\Delta_l)(D_k\Delta_m)^* + \text{c.c.}] \end{aligned} \tag{4.1}$$

with respect to one component of the order parameter Δ_s^*. This gives an equation of the form

$$\alpha\Delta_s = K^*_{kl;ss'}D_kD_l\Delta'_s - \tilde{\kappa}H\Delta_s \tag{4.2}$$

(note that we are ignoring the difference between h and H in this approximation). Since the gradient terms in

Eq.(4.2) cannot in general be reduced into a $D_x^2 + D_y^2$ form, the lowest Landau level wave functions are not sufficient for calculating the condensate wave function in general. However, the solution of this equation can still be expressed in terms of Landau level wave functions:

$$\begin{aligned}\phi_n(\mathbf{r}) &= [2^n \pi^{1/2}(n!)]^{-1/2} \\ &\times \sum_m q_m e^{ik_m x'} e^{-(y'-k_m)^2/2} H_n(y'-k_m),\end{aligned} \tag{4.3}$$

where H_n is the Hermite polynomial of nth order, x' and y' are x, y coordinates in the unit of the magnetic length $l = (\Phi_0/2\pi H)^{1/2}$. This is because D_i's can be expressed as a linear combination of the raising and lowering operators of the Landau levels $\Pi_\pm$, since $\Pi_\pm = l(D_x \pm iD_y)/\sqrt{2}$.

This wave function describes a vortex lattice when $|\phi_n(\mathbf{r})|$ is periodic in the lattice vectors $\mathbf{a_1} = al(1,0)$ and $\mathbf{a_2} = bl(\cos\theta, \sin\theta)$, and $\phi_n(\mathbf{r})$ vanish at $m_1\mathbf{a_1} + m_2\mathbf{a_1}$ when m_1 and m_2 are integers. This requires

$$\begin{aligned}k_m &= 2\pi(m-1/2)/a = (m-1/2)\sqrt{2\pi\sigma} \\ q_m &= e^{i\pi m(\varsigma+1-m\varsigma)},\end{aligned} \tag{4.4}$$

where $\sigma = (b/a)\sin\theta$ and $\varsigma = (b/a)\cos\theta$. Note that we used the flux quantization condition $ab\sin\theta = 2\pi$. For a lattice of HQV's we need to consider a second lattice that is translated by $l\boldsymbol{\tau} = l(\tau_x, \tau_y)$ with respect to the first lattice. For the wave function of this lattice, we can use[30]

$$\tilde{\phi}_n(\mathbf{r}) = e^{i\tau_y x}\phi_n(\mathbf{r} - \boldsymbol{\tau}), \tag{4.5}$$

the phase factor being chosen so that $\Pi_- \tilde{\phi}_0(\mathbf{r}) = 0$. This latter condition ensures that the translated eigenstates have the same gauge as the untranslated eigenstates.

The formalism considered here gives us not only the energy due to interaction between vortices but also the core energy of vortices as well. This is because our vortex lattice wave function gives full description of the core regions. From the linearized GL equation we use here, a full quantum vortex is merely two HQV's of opposite spins coinciding at a same point. This means that the full quantum vortex core energy, if we ignore the cross term between two spin components in $f^{(4)}_{hom}$, is approximately twice the core energy of a HQV. If the HQV core energy is actually larger than this, that would make stabilization of the HQV more difficult, i.e., the largest allowed value for ρ_{sp}/ρ_s for the HQV lattice would be smaller than what we obtain through the formalism used here.

The lattice structure can be determined by finding $(\sigma, \varsigma, \boldsymbol{\tau})$ that minimize the free energy expectation value. For this, we first set the amplitude of OP to minimize the energy for given $(\sigma, \varsigma, \boldsymbol{\tau})$ (the amplitude depends on $H_{c2} - H$), and then compare energy for different values of $(\sigma, \varsigma, \boldsymbol{\tau})$. To determine these structures we will need to take the spatial integral of the product of four Landau level wavefunctions. We have computed these integrals in the Appendix C.

V. LOWEST LANDAU LEVEL SOLUTION

In the bulk of this section we provide detailed analysis of the lowest Landau level solution for the non-chiral triplet superconductors and briefly comment on the chiral case in subsection V C. In this case the relevant free energy (for orthorhombic, tetragonal, and hexagonal materials) is

$$\begin{aligned}f &= \sum_{s=\uparrow\uparrow,\downarrow\downarrow}\left[-\alpha|\Delta_s|^2 + \beta_1|\Delta_s|^4/2 + \left(\sum_{i=x,y,z} K_i|D_i\Delta_s|^2\right)\right] \\ &- \beta_2|\Delta_{\uparrow\uparrow}|^2|\Delta_{\downarrow\downarrow}|^2 - \tilde{\kappa}h(|\Delta_{\uparrow\uparrow}|^2 - |\Delta_{\downarrow\downarrow}|^2) \\ &+ \gamma[\Delta^*_{\uparrow\uparrow}\Delta_{\downarrow\downarrow}(\mathbf{D}\Delta_{\uparrow\uparrow})\cdot(\mathbf{D}\Delta_{\uparrow\uparrow})^* + \text{c.c}] + \frac{h^2}{8\pi} - \frac{Hh}{4\pi}.\end{aligned} \tag{5.1}$$

As mentioned before, we assume that we have re-scaled lengths so that we can take $K_z = K_x = K_y = K$. First look at the upper critical field problem. The linearized GL equation is:

$$\frac{\alpha l^2}{K}\Delta_s = \left(1 + 2\Pi_+\Pi_- - s\frac{Hl^2}{K}\tilde{\kappa}\right)\Delta_s. \tag{5.2}$$

The largest H_{c2} occurs when the Δ_s are in the lowest Landau level. This leads to two possible values for the upper critical field,

$$H^{\pm}_{c2} = \frac{\alpha\Phi_0}{2\pi(K \pm Hl^2\tilde{\kappa})} \tag{5.3}$$

We assume that $\tilde{\kappa} > 0$ so that $\Delta_{\uparrow\uparrow}$ has the larger H_{c2}. We do not assume that the splitting between these two critical fields is large since it is given by the small Zeeman term $\tilde{\kappa}$.

Now consider the expectation value of f in terms of the Landau level wave functions. Applying Eq. (4.2) to the first two terms of Eq. (5.1) gives the lowest Landau level solutions $\Delta_{\uparrow\uparrow} = C_{\uparrow\uparrow}\phi_0$ and $\Delta_{\downarrow\downarrow} = C_{\downarrow\downarrow}e^{i2\alpha}\tilde{\phi}_0$. Inserting this solution as determined at $H = H_{c2}$ gives

$$\begin{aligned}\langle f\rangle &= -\frac{1}{c}\langle\mathbf{j}\cdot\delta\mathbf{A}\rangle + \beta_1\langle|\phi_0|^4\rangle(C^4_{\uparrow\uparrow} + C^4_{\downarrow\downarrow})/2 \\ &+ \left[\frac{2\gamma}{l^2}(\langle|\phi_0|^2|\tilde{\phi}_0|^2\rangle - \langle|\phi_0|^2|\tilde{\phi}_1|^2\rangle) - \beta_2\langle|\phi_0|^2|\tilde{\phi}_0|^2\rangle\right]C^2_{\uparrow\uparrow}C^2_{\downarrow\downarrow} \\ &+ \frac{\langle h_s^2\rangle}{8\pi} - \frac{H^2}{8\pi} + \alpha\left[1 - \frac{K + H_{c2}l^2\tilde{\kappa}}{K - H_{c2}l^2\tilde{\kappa}}\right]C^2_{\downarrow\downarrow},\end{aligned} \tag{5.4}$$

where $\mathbf{j}$ is the supercurrent, $\delta\mathbf{A}$ is deviation of the vector potential from what we would have for the $h = H_{c2}$, and h_s is the screening field of superconductor. Note that in this approximation $\mathbf{j}$ is calculated solely from quadratic terms, ignoring γ terms, and by the Maxwell equation $\nabla\times\boldsymbol{h_s} = 4\pi\boldsymbol{j}/c$. More specifically

$$\begin{aligned}\boldsymbol{j} = \; & 2eK[\Delta^*_{\uparrow\uparrow}(\boldsymbol{D}\Delta_{\uparrow\uparrow}) + \Delta^*_{\downarrow\downarrow}(\boldsymbol{D}\Delta_{\downarrow\downarrow}) + c.c] \\ & - c\tilde{\kappa}\nabla\times\hat{z}(|\Delta_{\uparrow\uparrow}|^2 - |\Delta_{\downarrow\downarrow}|^2)\end{aligned} \tag{5.5}$$

Since $\nabla\times\delta\boldsymbol{A}=\hat{\boldsymbol{z}}(h_s+H-H_{c2})$ and, from Maxwell's equations, $\nabla\times\boldsymbol{h_s}=4\pi\boldsymbol{j}/c$, partial integration leads to[31]

$$\begin{aligned}\frac{1}{c}\langle\mathbf{j}\cdot\delta\mathbf{A}\rangle &= \frac{1}{4\pi}\langle\boldsymbol{h_s}\cdot(\boldsymbol{h_s}+\boldsymbol{H}-\hat{\boldsymbol{z}}H_{c2})\rangle\\ &= \frac{\langle h_s^2\rangle}{4\pi}+\frac{H_{c2}-H}{4\pi}\langle h_s\rangle. \end{aligned}\tag{5.6}$$

Meanwhile, when we calculate the expectation value of γ, we set $h=H_{c2}$. This leads to the free energy of

$$\begin{aligned}\langle f\rangle &= -\frac{H_{c2}-H}{4\pi}\langle h_s\rangle-\frac{\langle h_s^2\rangle}{8\pi}-\frac{H^2}{8\pi}\\ &+ \alpha\left[1-\frac{K+H_{c2}l^2\tilde{\kappa}}{K-H_{c2}l^2\tilde{\kappa}}\right]C_{\downarrow\downarrow}^2+\frac{\beta}{2}\langle|\phi_0|^4\rangle(C_{\uparrow\uparrow}^4+C_{\downarrow\downarrow}^4)\\ &+ \left[\frac{2\gamma}{l^2}(\langle|\phi_0|^2|\tilde{\phi}_0|^2\rangle-\langle|\phi_0|^2|\tilde{\phi}_1|^2\rangle)-\beta_2\langle|\phi_0|^2|\tilde{\phi}_0|^2\rangle\right]C_{\uparrow\uparrow}^2C_{\downarrow\downarrow}^2\\ &\equiv -\frac{H^2}{8\pi}+\langle\tilde{f}\rangle. \end{aligned}\tag{5.7}$$

The screening field h_s can be calculated by assuming $|\hat{\Delta}|\propto(H_{c2}-H)^{1/2}$ near the second order phase transition at $H=H_{c2}$. Here we will deal only with $O(|\hat{\Delta}|^4)$ (or equivalently, $O(1-H/H_{c2})^2$) and ignore higher order terms. This allows us to calculate $\mathbf{j}$, and consequently h_s, solely from quadratic terms. In the lowest Landau level, this yields the screening field:

$$\begin{aligned}h_s &= \frac{8\pi^2K}{\Phi_0}(|\Delta_{\uparrow\uparrow}|^2+|\Delta_{\downarrow\downarrow}|^2)-4\pi\tilde{\kappa}(|\Delta_{\uparrow\uparrow}|^2-|\Delta_{\downarrow\downarrow}|^2)\\ &= \left(\frac{8\pi^2K}{\Phi_0}-4\pi\tilde{\kappa}\right)C_{\uparrow\uparrow}^2|\phi_0|^2+\left(\frac{8\pi^2K}{\Phi_0}+4\pi\tilde{\kappa}\right)C_{\downarrow\downarrow}^2|\tilde{\phi}_0|^2. \end{aligned}\tag{5.8}$$

Inserting Eq. (5.8) into Eq. (5.7), the free energy takes the following form:

$$\langle\tilde{f}\rangle=-\tilde{\alpha_1}C_{\uparrow\uparrow}^2-\tilde{\alpha_2}C_{\downarrow\downarrow}^2+\tilde{\beta}_1C_{\uparrow\uparrow}^4+\tilde{\beta}_2C_{\downarrow\downarrow}^4+\tilde{\beta}_3C_{\uparrow\uparrow}^2C_{\downarrow\downarrow}^2 \tag{5.9}$$

with terms quadratic or quartic in $C_{\uparrow\uparrow}$ and $C_{\downarrow\downarrow}$ with coefficients that are independent in general (we will further specify these coefficients is the next two subsections). In the absence of screening fields, Zeeman-fields and the term proportional to γ, the form of the free energy in Eq. (5.7) is similar to that examined in Ref. 24 in the context of two-component Bose condensates (spin-half spinor condensate).In that case, the vortex lattice structure is solely determined by the competition between the β_1 and β_2 terms of Eq. (5.1); the β_1 term determines the interaction energy within each vortex lattices, and the β_1 term the interaction energy between two fractional vortex species each forming lattices. Specifically, the quartic term $-\beta_2\langle|\phi_0|^2|\tilde{\phi}|^2\rangle C_{\uparrow\uparrow}^2C_{\downarrow\downarrow}^2$ determines the stability of the HQV lattice. If $\beta_2<0$, then a HQV lattice is the ground state. If $\beta_2>0$, then full quantum vortex lattice is the ground state. In Appendix A, we show that $\beta_2=0$ in weak-coupling theories, so that the two lattice structures are degenerate. In the rest of the paper, we will focus on aspects that are unique to triplet superconductors in subsections V A and V B: the effects of the screening fields $f_{in}^{(4)}$, and the Zeeman-field.

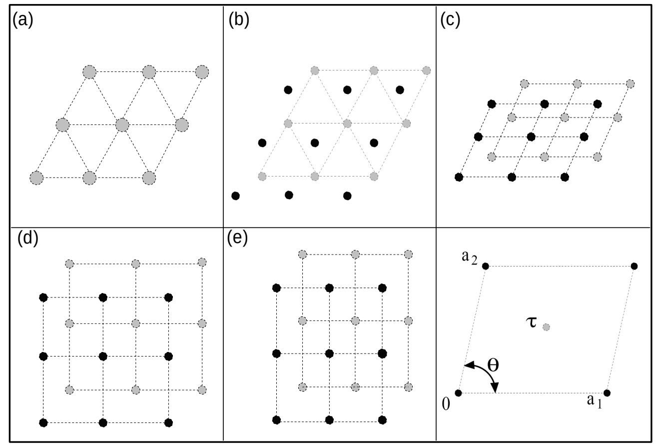

FIG. 1: Real space vortex lattice structure

A. The effects of screening and $f_{in}^{(4)}$

Here we look into the effect of screening. Ignoring the Zeeman field (in weak-coupling theories, the Zeeman field is vanishing in the clean limit), we have symmetry between $C_{\uparrow\uparrow}$ and $C_{\downarrow\downarrow}$, giving us

$$\tilde{\alpha_1}=\tilde{\alpha_2}\equiv\tilde{\alpha}=\frac{2\pi K(H_{c2}-H)}{\Phi_0}\langle|\phi_0|^2\rangle \tag{5.10}$$

and

$$\tilde{\beta}_1=\tilde{\beta}_2\equiv\tilde{\beta}=\left(\frac{\beta}{2}-\frac{8\pi^3K^2}{\Phi_0^2}\right)\langle|\phi_0|^4\rangle, \tag{5.11}$$

which means that the free energy in Eq.(5.9) takes a simpler form:

$$\langle\tilde{f}\rangle=-\tilde{\alpha}(C_{\uparrow\uparrow}^2+C_{\downarrow\downarrow}^2)+\tilde{\beta}(C_{\uparrow\uparrow}^4+C_{\downarrow\downarrow}^4)+\tilde{\beta}_3C_{\uparrow\uparrow}^2C_{\downarrow\downarrow}^2, \tag{5.12}$$

with

$$\tilde{\beta}_3=\frac{2\gamma}{l^2}\langle|\phi_0|^2(|\tilde{\phi}_0|^2-|\tilde{\phi}_1|^2)\rangle-\left(\frac{16\pi^3K^2}{\Phi_0^2}+\beta_2\right)\langle|\phi_0|^2|\tilde{\phi}_0|^2\rangle. \tag{5.13}$$

Now the free energy of Eq.(5.7) can be minimized by choosing $C_{\uparrow\uparrow}^2=C_{\downarrow\downarrow}^2=\tilde{\alpha}/(2\tilde{\beta}+\tilde{\beta}_3)$ (which gives $|\hat{\Delta}|\propto(H_{c2}-H)^{1/2}$ as mentioned), giving us the free energy expectation value

$$\langle f\rangle=-\frac{H^2}{8\pi}-\frac{\tilde{\alpha}^2}{2\tilde{\beta}+\tilde{\beta}_3}. \tag{5.14}$$

Eqs.(5.13) and (5.14) allows for understanding the role of both screening and $f_{in}^{(4)}$. The screening affect the vortex lattice structure through the dependence of terms proportional to K^2 in Eqs. (5.11) and (5.13) on the lattice structure parameters ς, σ and $\boldsymbol{\tau}$. Since all quartic expectation values depend on the lattice structure, the lattice structure will be determined through minimizing $(2\tilde{\beta}+\tilde{\beta}_3)$. Since the magnitude of K^2 term in Eq.(5.13)

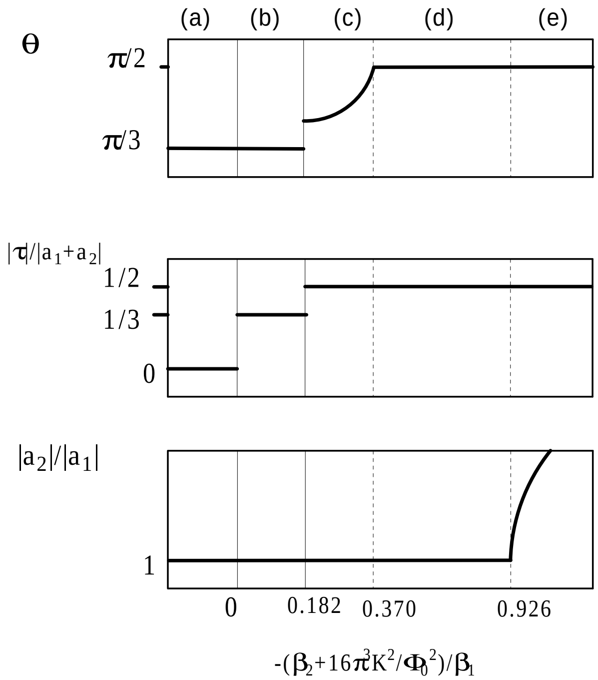

FIG. 2: Transition between different vortex lattice structures when $\gamma = 0$. (a)-(e) label the vortex lattice structure shown in Fig. (1).

is larger for full quantum vortices (Eq.(5.11) is not affected), screening tend to disfavor HQV lattices, in line with earlier observation for isolated HQV's[3]. However, since weak-coupling theories lie near the point $\tilde{\beta}_3 = 0$ we expect the screening effect will put the physical system at a fine balance between interlacing lattices of HQV's and the ordinary Abrikosov vortex lattice. Hence it should be possible to observe the transition between the two phases upon small change of field and temperature. Indeed, the interaction $f_{in}^{(4)}$ plays this role. In particular, we numerically find that for $\gamma > 0$, then this term tends to favor HQV lattices. A positive sign of γ occurs when ρ_{sp}, ρ_s and this is to be expected in spin-triplet superconductors[3]. Note, that unlike the other contributions in $\tilde{\beta}_3$, the contribution from $f_{in}^{(4)}$ vanishes as $T \to T_c$. Consequently, this term can drive a field and temperature dependent transition between a HQV lattice and a full quantum lattice. Figures 1 and 2 show the phase diagram for $\gamma = 0$. Note the similarity between the calculated phase diagram and that found in the context of two-component Bose condensates[24,32].

B. The effects of the Zeeman term

For simplicity, we consider here the effect of Zeeman field alone ignoring screening and setting $\gamma = 0$ (ignoring $f_{in}^{(4)}$). In the presence of Zeeman field, the free energy Eq. (5.9) would have different coefficients for two quartic terms:

$$\langle \tilde{f} \rangle = -\tilde{\alpha_1} C_{\uparrow\uparrow}^2 - \tilde{\alpha_2} C_{\downarrow\downarrow}^2 + \tilde{\beta}(C_{\uparrow\uparrow}^4 + C_{\downarrow\downarrow}^4) + \tilde{\beta}_3 C_{\uparrow\uparrow}^2 C_{\downarrow\downarrow}^2 \quad (5.15)$$

where $\tilde{\alpha}_1 = \alpha + \frac{K}{l^2} + H\tilde{\kappa}$, $\tilde{\alpha}_2 = \alpha + \frac{K}{l^2} - H\tilde{\kappa}$, $\tilde{\beta} = \beta_1 \langle |\phi_0|^4 \rangle$, and $\tilde{\beta}_3 = -\beta_2 \langle |\phi_0|^2 |\tilde{\phi}_0|^2|^2 \rangle$. The Zeeman field has two main consequences: (i) it typically leads to two phase transitions. In the first phase $C_{\uparrow\uparrow} \neq 0$ and $C_{\downarrow\downarrow} = 0$ and in the second phase, both components are non-zero. The first phase is analogous to the ^{3}He A_1 phase, with a non-unitary spin-triplet order parameter. However, weak-coupling theories prefer unitary spin-triplet states and this drives the second transition. (ii) In the fractional vortex lattice phase where both components are non-zero, the magnetic flux contained by isolated fractional vortices is no longer a half-integral flux quanta. Instead, two types of vortices each carry fractional flux values of

$$\Phi_i = \Phi_0 \frac{|c_i|^2}{|c_1|^2 + |c_2|^2}. \quad (5.16)$$

The double transition is possible if $2\tilde{\beta} + \tilde{\beta}_3 > 0$ (note again that weak-coupling theories yield $\tilde{\beta}_3 = 0$ and $\tilde{\beta} > 0$), there can be two transitions with a second transition appearing at a temperature

$$T_{c2} - T_{c1} = \frac{4\tilde{\beta}}{2\tilde{\beta} + \tilde{\beta}_3} \frac{K + H_{c2} l^2 \tilde{\kappa}}{K_1 - H_{c2} l^2 \tilde{\kappa}} T_{c1}. \quad (5.17)$$

In the high temperature phase, the vortex lattice is hexagonal and, at the second transition, the lattice will remain hexagonal and the second component will either coincide with first or be displaced half a hexagonal vortex lattice vector from the first. As temperature is further reduced below the second transition, the lattice will continuously deform, asymptotically approaching the phases presented in the subsection V A (those shown in Fig. 2). The resulting phase diagram is qualitatively shown in Fig. 3.

Both consequences of the Zeeman field stem from breaking the additional $\mathbb{Z}_2$ symmetry that is present when $\tilde{\alpha}_1 = \tilde{\alpha}_2$. In general, the existence of fractional vortices is the result of the $U(1) \times U(1)$ symmetry of the free energy. When there is additional $\mathbb{Z}_2$ symmetry due to $\tilde{\alpha}_1 = \tilde{\alpha}_2$, the flux contained in each fractional vortices are restricted to be half the flux quantum since the two components of the order parameter are no longer degenerate in a magnetic field. In Section VII, we will see that this helps us distinguish a lattice of HQVs from a lattice of full quantum vortex.

C. Chiral Triplet superconductors: lowest Landau level solution

The chiral triplet superconductor with tetragonal symmetry, because of the inhomogeneous quadratic terms we

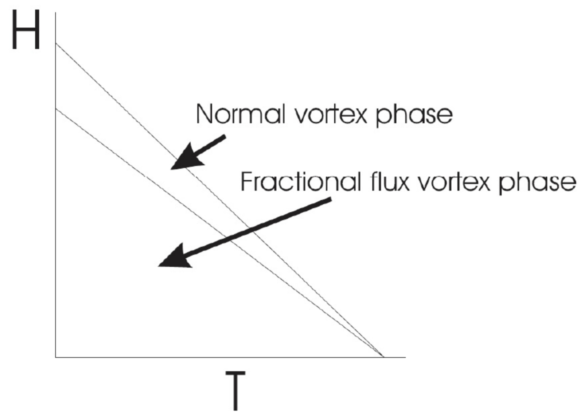

FIG. 3: The role of the Zeeman field is to cause two transitions and to change the HQV lattice phase to a a phase with two types of fractional vortices in which the two fractional fluxes sum to Φ_0. Far below the second transition it is expected that these fractions will be well approximated by 1/2.

have already seen in Eq. (3.12),

$$\begin{aligned} f_{in}^{(2,s)} &= K_1(|\mathbf{D}\Delta_{s,+}|^2 + |\mathbf{D}\Delta_{s,-}|^2) \\ &+ K_2[\{(D_x\Delta_{s,+})(D_x\Delta_{s,-})^* - (D_y\Delta_{s,+})(D_y\Delta_{s,-})^*\}/2 \\ &+ \{(D_x\Delta_{s,-})(D_x\Delta_{s,+})^* - (D_y\Delta_{s,-})(D_y\Delta_{s,+})^*\}/2 \\ &+ i\{(D_x\Delta_{s,-})(D_y\Delta_{s,+})^* + (D_y\Delta_{s,-})(D_x\Delta_{s,+})^*\}/2 \\ &- i\{(D_x\Delta_{s,+})(D_y\Delta_{s,-})^* + (D_y\Delta_{s,+})(D_x\Delta_{s,-})^*\}/2] \\ &+ K_4(|D_z\Delta_{s,+}|^2 + |D_z\Delta_{s,-}|^2), \end{aligned} \tag{5.18}$$

has a much more complicated quadratic free energy,

$$f_0^{(2)} = \sum_{s=\Uparrow,\Downarrow} [-\alpha(|\Delta_{s,+}|^2 + |\Delta_{s,-}|^2) + f_{in}^{(2,s)}], \tag{5.19}$$

even when we exclude the Zeeman field and any spin-orbit coupling.

Due to these inhomogenous quadratic terms, we cannot put both chirality components in the lowest Landau level. This is due to the presence of $[(D_x\Delta_{s,\sigma})(D_y\Delta_{s,-\sigma})^* + \text{c.c}]$ terms in $f_{in}^{(2,s)}$. The above quadratic free energy of Eq. (5.19), together with Eq. (3.13) that gives us the energy splitting between two chiralities, leads to the linearized GL equation

$$\alpha l^2 \begin{pmatrix} \Delta_{s+} \\ \Delta_{s-} \end{pmatrix} = \begin{bmatrix} K_1(1+2\Pi_+\Pi_-) - \delta K & K_2\Pi_-^2 \\ K_2\Pi_+^2 & K_1(1+2\Pi_+\Pi_-) + \delta K \end{bmatrix} \begin{pmatrix} \Delta_{s+} \\ \Delta_{s-} \end{pmatrix}. \tag{5.20}$$

However, a lowest Landau level solution that satisfies Eq. (5.20) may still have the highest H_{c2} and therefore be possible[15]. This would lead to a nonzero order parameter for only one chirality - $(\Delta_{s+}, \Delta_{s-}) = C(0, \phi_0)$ - and requires $\delta K < -\frac{K_2^2}{4K_1}$. In this case, the vortex energetics of the chiral triplet superconductor is identical to that of the nonchiral triplet superconductor, for inserting this lowest Landau level solution into the full Gibbs free energy leads us back to Eq. (5.4).

However, Eq. (5.20) can also give us a solution with Landau level mixing; in this case, both Δ_{s+} and Δ_{s-} are nonzero. We will present discussion on this Landau level mixing in Appendix B.

VI. ROLE OF SPIN-ORBIT COUPLING

A. Generic Case

If the material has orthogonal or tetragonal symmetry (though not necessarily true for hexagonal symmetry, which is discussed in the next subsection), then there will exist spin-orbit coupling terms of the type

$$\epsilon\Delta_{\Uparrow\Uparrow}\ \Delta_{\Downarrow\Downarrow}^*. \tag{6.1}$$

Such terms break the $U(1) \times U(1)$ symmetry and consequently isolated fractional flux vortices are no longer stable. Nevertheless, a fractional flux quantum vortex lattice can still exist, provided that the separation between vortices is less than ξ_{so} defined through $\xi_{so}^2 = K/\epsilon$.

For completeness, we write here the spin-orbit coupling terms that appear in the context of a chiral spin-triplet superconductor. While we do not include these terms in calculations, they may be useful in other contexts. Due to the tetragonal C_4 symmetry, homogeneous spin-orbit coupling terms should be invariant under the transformation

$$(\Delta_{\Uparrow,+}, \Delta_{\Uparrow,-}, \Delta_{\Downarrow,+}, \Delta_{\Downarrow,-}) \to (-\Delta_{\Uparrow,+}, \Delta_{\Uparrow,-}, \Delta_{\Downarrow,+}, -\Delta_{\Downarrow,-}).$$

To the quadratic order, this condition is satisfied by[34]

$$\begin{aligned} f_{SO}^{(2)} &= \epsilon_1(|\Delta_{\Uparrow,+}|^2 + |\Delta_{\Downarrow,-}|^2 - |\Delta_{\Uparrow,-}|^2 - |\Delta_{\Downarrow,+}|^2) \\ &+ \epsilon_2[(\Delta_{\Uparrow,-})(\Delta_{\Downarrow,+})^* + \text{c.c.}] \\ &+ \epsilon_3[(\Delta_{\Uparrow,+})(\Delta_{\Downarrow,-})^* + \text{c.c.}]. \end{aligned} \tag{6.2}$$

We note here that ϵ_i's can be estimated from the recent ARPES data[20,21].

B. Hexagonal Materials

For hexagonal materials, there exist spin-triplet pairing states for which no such terms such as that in the above equation appear. These states belong to the two-dimensional representations labelled Γ_5^- and Γ_6^- in the review article by Sigrist and Ueda[25]. Consequently, these materials need to be considered more carefully.

We will now show that in hexagonal materials, a little away from H_{c2}, spin-orbit coupling does not break $U(1) \times U(1)$ symmetry. For hexagonal materials, the only term that exists in the GL free energy that is due to spin-orbit coupling is (note that the inclusion of this term gives rise the complete free energy found that is found in Sigrist and Ueda for the $\Gamma_{5,6}^-$ representations):

$$\begin{aligned} f_{SO} &= K_{so}\Big\{(D_x\Delta_{\downarrow\downarrow})(D_x\Delta_{\uparrow\uparrow})^* - (D_y\Delta_{\downarrow\downarrow})(D_y\Delta_{\uparrow\uparrow})^* \\ &+ (D_x\Delta_{\uparrow\uparrow})(D_x\Delta_{\downarrow\downarrow})^* - (D_y\Delta_{\downarrow\downarrow})(D_y\Delta_{\uparrow\uparrow})^* \\ &- i[(D_x\Delta_{\downarrow\downarrow})(D_y\Delta_{\uparrow\uparrow})^* + (D_y\Delta_{\downarrow\downarrow})(D_x\Delta_{\uparrow\uparrow})^*] \\ &+ i[(D_x\Delta_{\uparrow\uparrow})(D_y\Delta_{\downarrow\downarrow})^* + (D_y\Delta_{\uparrow\uparrow})(D_x\Delta_{\downarrow\downarrow})^*]\Big\}/2 \end{aligned} \tag{6.3}$$

With the field along the c-axis, the solution to the quadratic problem satisfies

$$\frac{\alpha l^2}{K}\begin{pmatrix}\Delta_{\uparrow\uparrow}\\ \Delta_{\downarrow\downarrow}\end{pmatrix} = \begin{pmatrix} 1+2N-K_z & \tilde{K}_{so}\Pi_-^2 \\ \tilde{K}_{so}\Pi_+^2 & 1+2N+K_z\end{pmatrix}\begin{pmatrix}\Delta_{\uparrow\uparrow}\\ \Delta_{\downarrow\downarrow}\end{pmatrix}$$

where $K_z = \frac{\tilde{K}Hl^2}{K}$ and $\tilde{K}_{so} = K_{so}/K$. All the eigenstates for this problem can be found analytically[30]. Typically, $|\tilde{K}_{so}| << 1$, so we will be interested in the eigenstates that contain the lowest Landau level (which will minimize the free energy when $\tilde{K}_{so} = 0$). The two relevant eigenstates that we wish to keep are: $(\Delta_{\uparrow\uparrow}, \Delta_{\downarrow,\downarrow}) = (\phi_0, \epsilon\phi_2)$ and $(\Delta_{\uparrow\uparrow}, \Delta_{\downarrow,\downarrow}) = (0, \phi_0)$ where ϵ is proportional to $\tilde{K}_{so}$. Note that unlike in the subsection V C we can keep both solutions because while we examined $H \sim H_{c2}$ in that subsection, we are a little away from H_{c2} in this subsection. We therefore write $\Delta_{\uparrow\uparrow} = \gamma_1\phi_0$ and $\Delta_{\downarrow\downarrow} = \gamma_1\epsilon\phi_2 + \gamma_2\tilde{\phi}_0$ to include these two eigenstates. For simplicity, we ignore screening and the Zeeman field to find the following free energy

$$\begin{aligned} \langle f\rangle = &-(1-H/H_{c2,1})|\gamma_1|^2 - (1-H/H_{c2,2})|\gamma_2|^2 \\ &+\beta_1[|\gamma_1|^4\langle|\phi_0|^4\rangle + \langle|\gamma_1\epsilon\phi_2 + \gamma_2\tilde{\phi}_0|^4\rangle] \\ &-\beta_2\langle|\gamma_1\phi_0|^2|\gamma_1\epsilon\phi_2+\gamma_2\tilde{\phi}_0|^2\rangle \end{aligned} \tag{6.4}$$

where $H_{c2,i}$ $(i = 1, 2)$ is the upper critical field for eigenstate i. Since spin-orbit coupling is expected to be small, this implies that $\epsilon << 1$, so keeping to linear order in ϵ yields:

$$\begin{aligned} \langle f\rangle = &-(1-H/H_{c2,1})|\gamma_1|^2 - (1-H/H_{c2,2})|\gamma_2|^2 \\ &+\beta_1\langle|\phi_0|^4\rangle(|\gamma_1|^4+|\gamma_2|^4) + \beta_2\langle|\phi_0|^2|\tilde{\phi}_0|^2\rangle|\gamma_1|^2|\gamma_2|^2 \\ &+\epsilon[\beta_1|\gamma_2|^2\gamma_2\gamma_1^*\langle|\tilde{\phi}_0|^2\tilde{\phi}_0\phi_2^*\rangle + \text{c.c}] \\ &-\epsilon[\beta_2|\gamma_1|^2\gamma_1\gamma_2^*\langle|\phi_0|^2\phi_2\tilde{\phi}_0^*\rangle + \text{c.c}] \end{aligned} \tag{6.5}$$

Without the last two terms, this theory is the same as that found for non-chiral spin-triplet superconductors with a Zeeman field but without any spin-orbit coupling. At the upper critical field, one of the two components γ_1 or γ_2 order and the vortex lattice will be hexagonal (this conclusion is correct even when including terms that are second order in ϵ). As the temperature or magnetic field is reduced, the last two terms in Eq. (6.5) can play an important role. These two terms break the $U(1) \times U(1)$ symmetry of the theory and therefore will tend to remove any HQV lattice phases. However, the spatial averages $\langle|\tilde{\phi}_0|^2\tilde{\phi}_0\phi_2^*\rangle$ and $\langle|\phi_0|^2\phi_2\tilde{\phi}_0^*\rangle$ *vanish* for a hexagonal vortex lattice (loosely speaking, this follows from noting that ϕ_n picks up a factor $e^{in\phi}$ under a rotation about $\hat{z}$ and that a hexagonal vortex lattice is symmetric under rotations of $\pi/3$) . The hexagonal symmetry of the materials conspires to remove this form of $U(1) \times U(1)$ symmetry breaking and the HQV lattice structures are still possible (indeed the theory is the same as that given for the non-chiral spin-triplet superconductors with a Zeeman field, but without spin-orbit coupling). Note that if $\boldsymbol{\tau} \neq 0$ (signaling the existence of the fractional vortex lattice), then $\langle|\tilde{\phi}_0|^2\tilde{\phi}_0\phi_2^*\rangle = \langle|\phi_0|^2\phi_2\tilde{\phi}_0^*\rangle = 0$ for any lattice geometry. Consequently, the last two terms of Eq. (6.5)) do not play any role in the theory of the fractional vortex lattices.

It is reasonable to ask if there are any other $U(1)\times U(1)$ symmetry breaking terms that we have neglected in the above analysis. Indeed there is one that appears at order ϵ^2: $\epsilon^2\beta_1\gamma_1^2(\gamma_2^2)^*\langle\phi_2^2(\tilde{\phi}_0^2)^*\rangle$. This term allows for the existence of a fractional vortex lattice phase subject to the constraint that $\boldsymbol{\tau}$ is half a vortex lattice translation vector[30]. There are also $U(1) \times U(1)$ that appear at order ϵ^3, but these vanish for the same reason as the order ϵ term. Consequently, the spin-orbit coupling for the hexagonal two-dimensional representations plays essentially the same role as the Zeeman field.

VII. OBSERVATION OF THE VORTEX LATTICE

The best way to determine both the vortex lattice structure and the vortex type is to observe the magnetic field distribution through the small angle neutron scattering. What we will see in this experiment is the Fourier transform $f(\mathbf{G})$ of the screening field of Eq. (5.8),

$$\begin{aligned} h_s(\mathbf{r}) &= (\frac{8\pi^2K}{\Phi_0} - 4\pi\tilde{\kappa})C_{\uparrow\uparrow}^2|\phi_0(\mathbf{r})|^2 \\ &+ (\frac{8\pi^2K}{\Phi_0} + 4\pi\tilde{\kappa})C_{\downarrow\downarrow}^2|\tilde{\phi}_0(\mathbf{r})|^2 \end{aligned} \tag{7.1}$$

- that is $h_s(\mathbf{r}) = \sum_{\mathbf{G}} f(\mathbf{G})\exp(i\mathbf{G}\cdot\mathbf{r})$, where $\mathbf{G}$ is the reciprocal lattice vectors of the vortex lattice in the unit of the inverse magnetic length.

The characteristic feature of the vortex lattice with half-vortices in the small angle neutron scattering experiment is the modulation of the Bragg peaks. The form factor of the Bragg peaks would be the $f(\mathbf{G})$ of the last paragraph. Using

$$|\phi_0(\mathbf{r})|^2 = \sum_{\mathbf{G}}(-1)^{m_1+m_2+m_1m_2}e^{-\mathbf{G}^2/2} \tag{7.2}$$

where $\mathbf{G} = m_1\mathbf{G_1} + m_2\mathbf{G_2}$, and $\mathbf{G_i}$'s are the basis vector

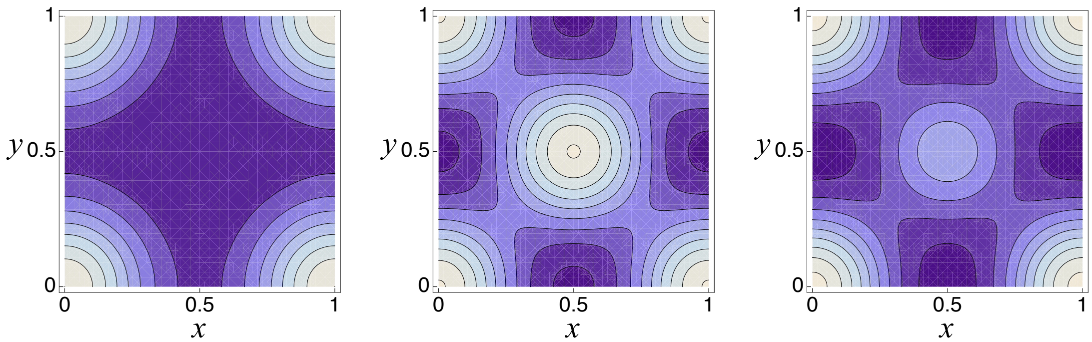

FIG. 4: The contour plots of the screening field in the real (that is, position) space for different types of vortex lattices: (a) a lattice of ordinary Abrokosov (full quantum) vortices (b) a HQV lattice (c) a lattice of fractional vortices in the presence of Zeeman field. The position is measured in units of magnetic length. Note the halving of the unit cell in going from the HQV lattice to the full quantum vortex lattice. When the Zeeman field is added in, the periodicity is that of the full quantum lattice, though the vortex lattice unit cell now has additional structure due to appearance of fractional flux at both the corners and center of the unit cell.

of the reciprocal lattice, we obtain the form factor

$$f(\boldsymbol{G}) = (-1)^{m_1+m_2+m_1 m_2} e^{-\boldsymbol{G}^2/2}[(\frac{8\pi^2 K}{\Phi_0} - 4\pi\tilde{\kappa})|C_{\uparrow\uparrow}|^2 + (\frac{8\pi^2 K}{\Phi_0} + 4\pi\tilde{\kappa})|C_{\downarrow\downarrow}|^2 e^{i\boldsymbol{G}\cdot\boldsymbol{\tau}}]. \quad (7.3)$$

This equation implies that the intensity $|f(\boldsymbol{G})|^2$ for our Bragg peaks does not come out same for all $\mathbf{G}$'s. This is because for almost all vortex lattice structure (the single exception being not very robuts honeycomb lattice) $\boldsymbol{\tau}$ is half a vortex translation vector so that we have $e^{i\boldsymbol{G}\cdot\boldsymbol{\tau}} = -1$ for half of $\mathbf{G}$'s and $e^{i\boldsymbol{G}\cdot\boldsymbol{\tau}} = 1$ for the other half. When there is no Zeeman field, $e^{i\boldsymbol{G}\cdot\boldsymbol{\tau}} = -1$ peaks disappear completely; natural given that magnetic field cannot distinguish the spin up and the spin down HQV's at all and thus sees the unit lattice vector halved. However, when the Zeeman field breaks down the $\mathbb{Z}_2$ symmetry between the spin up-up pairs and down-down pairs, we now see a secondary peak for $e^{i\boldsymbol{G}\cdot\boldsymbol{\tau}} = -1$ as shown on Fig. (4).

Another promising direction for detecting fractional vortex lattice would be to use spin-polarized STM to probe the vortex cores. The key point is that the low energy quasi-particle spins have opposite polarization in the two different HQV's. This is because for half of HQV cores, we have $\Delta_{\uparrow\uparrow} = 0$ and $\Delta_{\downarrow\downarrow} \neq 0$, so that only spin-down quasi-particles are gapped. On the other hand, for the other half of HQV cores, only spin-up quasi-particles are gapped. This spin-polarization of the subgap core modes should be readily detected through spin-polarized STM.

VIII. CONCLUSION

In this paper we explored various possibilities for fractional vortex lattice structures in spin triplet superconductors starting from the most general from of Gibbs free energy that is allowed by the symmetry of the order parameter and that of the lattice symmetries relevant for three candidate spin triplet superconductors, namely single layer ruthenate Sr_2RuO4[11,12] cobaltate $Na_xCoO_2 \cdot yH_2O$[13] and organic[14] $(TMTSF)_2ClO_4$. The focus of our analysis was on the role of aspects unique to triplet superconductors, such as (i) Cooper pair Zeeman field, (ii) spin-orbit coupling, (iii) screening, and (iv) interaction effects in the energetics of the vortex lattice structure. (i) The Cooper pair Zeeman field breaks a $\mathbb{Z}_2$ symmetry of the free energy whose presence constrains the fractional vortices to contain half integral flux quanta. The resulting structure is that of two interlacing lattices of vortices containing arbitrary fraction of flux quanta that adds up to one flux quanta. Such fractional vortex lattices will have interesting field distributions in vortex lattice unit cell due to internal structures within the unit cell. (ii) The effect of spin-orbit coupling is lattice symmetry specific. In hexagonal lattices systems such as cobaltates $Na_xCoO_2{\cdot}yH_2O$, spin-orbit coupling has the same effect as the Cooper pair Zeeman field, supporting fractional vortex lattices. However, for tetragonal or orthorhombic lattices, sufficiently strong spin-orbit coupling generally favors ordinary Abrikosov vortex lattice over HQV's. However, such an effect is relatively mild in a dense vortex lattice, provided that the separation between the HQV vortices is less that a length set by the spin-orbit coupling. (iii) The Meissner screening effectively generates attraction between two HQV's with opposite winding of the spin phase and weakly destablizes the HQV's. (iv) The interaction effects clearly support energetic stability of HQV's within the GL theory. The interaction effects represented by inhomogeneous (unique to triplet superconductors) quartic terms can drive difference in effective superfluid stiffness $\rho_{sp} < \rho_s$ which stabilizes HQV's in the London limit. When the above effects are put together, all weak coupling theories we examined appears to lie at the point of fine balance between ordinary Abrikosov vortex lattice and lattices of HQV's. Hence it should be possible to observe transitions between these structures with small changes of parameters. This further motivates experimental search for these fractional vortex lattices. We have sketched possible routes for such searches using neutron scattering or spin polarized STM.

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APPENDIX A: GINZBURG LANDAU ENERGY: FOURTH ORDER TERMS FROM WEAK-COUPLING THEORY

The GL free energy can be determined in the weak-coupling limit. In the context of the existence of 1/2 qv lattice structures, the result for the fourth order terms in the free energy turns out to be highly relevant. As shown here, this reveals that weak-coupling theories sit at a point in which the 1/2 qv and the full qv lattices are degenerate. This indicates that interactions beyond the weak-coupling limit are essential to determining the which lattice structure actually appears (screening plays a role here as well as shown earlier).

The portion of the free energy we calculate here is given in Eq. 3.6

$$f^{(4)}_{hom} = \beta_1 (\sum_i |\Delta_i|^2)^2 + \beta_2 |\Delta_{\uparrow\uparrow}|^2 |\Delta_{\downarrow\downarrow}|^2. \tag{A1}$$

The weak-coupling limit (without spin-orbit coupling) yields (this follows from Ref. 25)

$$f^{(4)}_{hom} \propto \langle |\mathbf{d}(\mathbf{k})|^4 \rangle + \langle \mathbf{q}^2(\mathbf{k}) \rangle \tag{A2}$$

where $\mathbf{q}(\mathbf{k}) = i\mathbf{d}(\mathbf{k}) \times \mathbf{d}^*(\mathbf{k})$, $\langle h(\mathbf{k})\rangle$ means average $h(\mathbf{k})$ over all $\mathbf{k}$ on the Fermi surface, and the proportionality constant can be found but it is not important for our considerations. When $\mathbf{q}$ is non-zero, then the superconducting state is called non-unitary. In weak-coupling theories, non-unitary states cost energy and typically do not appear. Using the gap structure of Eq. 2.2, we find

$$\begin{aligned} f^{(4)}_{hom} &\propto \langle |f(\mathbf{k})|^4 \rangle [(|\Delta_{\uparrow\uparrow}|^2 + |\Delta_{\downarrow\downarrow}|^2)^2 + (|\Delta_{\uparrow\uparrow}|^2 - |\Delta_{\downarrow\downarrow}|^2)] \\ &= 2\langle |f(\mathbf{k})|^4 \rangle (|\Delta_{\uparrow\uparrow}|^4 + |\Delta_{\downarrow\downarrow}|^4). \end{aligned} \tag{A3}$$

This implies that $\beta_2 = -2\beta_1$, independent of the shape of the Fermi surface. The lack of interaction between the two components of the gap function leads to the degeneracy between the 1/2 qv and the full qv lattice structures.

APPENDIX B: RUTHENATE - THE LANDAU LEVEL MIXING

We show here how we can have the Landau level mixing in a chiral triplet superconductor. The case we are considering here is in the weak pairing regime and has tetragonal crystalline symmetry and a cylindrical Fermi surface. Let us consider again the linearized GL equation:

$$l^2 \begin{pmatrix} \Delta_{s+} \\ \Delta_{s-} \end{pmatrix} = \frac{K}{\alpha} \begin{pmatrix} 1 + 2\Pi_+\Pi_- & \Pi_-^2 \\ \Pi_+^2 & 1 + 2\Pi_+\Pi_- \end{pmatrix} \begin{pmatrix} \Delta_{s+} \\ \Delta_{s-} \end{pmatrix}, \tag{B1}$$

where $s = \uparrow\uparrow, \downarrow\downarrow$. (Note that, though otherwise same as Eq. (5.20), we now ignore the energy splitting between the $\pm$ chiralities and set $K_1 = K_2 = K$.) This matrix equation as a solution in the form

$$\begin{pmatrix} \Delta_{s+} \\ \Delta_{s-} \end{pmatrix} = C_s \begin{pmatrix} \phi_0 \\ -\delta\phi_2 \end{pmatrix}, \tag{B2}$$

where $\delta = \sqrt{3} - \sqrt{2}$.

When we ignore the Zeeman field, much of the vortex lattice energetics of the lowest Landau level case remains valid with the Landau level mixing. For instance, the two main formulas of Section V. A, Eqs.(5.12),

$$\langle \tilde{f} \rangle = -\tilde{\alpha}(C^2_{\uparrow\uparrow} + C^2_{\downarrow\downarrow}) + \tilde{\beta}(C^4_{\uparrow\uparrow} + C^4_{\downarrow\downarrow}) + \tilde{\beta}_3 C^2_{\uparrow\uparrow} C^2_{\downarrow\downarrow}, \tag{B3}$$

and (5.14)),

$$\langle f \rangle = -\frac{H^2}{8\pi} - \frac{\tilde{\alpha}^2}{2\tilde{\beta} + \tilde{\beta}_3}, \tag{B4}$$

remains valid, mainly due to $|\hat{\Delta}| \propto (H_{c2} - H)^{1/2}$. This means we can still calculate h_s, solely from quadratic terms. For quadratic terms, we simply have two copies (for $s = \uparrow\uparrow$ and $\downarrow\downarrow$) of what was obtained for the case of $\mathbf{d} = (k_x + ik_y)\hat{\mathbf{z}}$ by one of us[18], we can use the formula for h_s for that case:

$$\begin{aligned} h_s &= \frac{8\pi^2 K}{\Phi_0} [C^2_{\uparrow\uparrow} \{(1 - 3\delta/\sqrt{2} + 2\delta^2)|\phi_0|^2 \\ &+ (2\delta^2 - \delta/\sqrt{2})|\phi_1|^2 + \delta^2 |\phi_2|^2\} \\ &+ C^2_{\downarrow\downarrow} \{(1 - 3\delta/\sqrt{2} + 2\delta^2)|\tilde{\phi}_0|^2 \\ &+ (2\delta^2 - \delta/\sqrt{2})|\tilde{\phi}_1|^2 + \delta^2 |\tilde{\phi}_2|^2\}]. \end{aligned} \tag{B5}$$

The spatial average of this equation is still proportional to $(C^2_{\uparrow\uparrow} + C^2_{\downarrow\downarrow})$ just like Eq. (5.8). Also, $\tilde{\alpha} \propto (H_{c2} - H)$ still stands:

$$\begin{aligned} \tilde{\alpha} &= \frac{2\pi K (H_{c2} - H)}{\Phi_0} [(1 - 3\delta/\sqrt{2} + 2\delta^2)\langle |\phi_0|^2 \rangle \\ &+ (2\delta^2 - \delta/\sqrt{2})\langle |\phi_1|^2 \rangle + \delta^2 \langle |\phi_2|^2 \rangle]. \end{aligned} \tag{B6}$$

However, the formula for $2\tilde{\beta}+\tilde{\beta}_3$ are much more complicated here, especially when we include all terms of Eqs. (3.14) and (3.15) for these coefficients. For sake of convenience, instead of directly writing down $\tilde{\beta}$ and $\tilde{\beta}_3$, we will list $\overline{h_s^2}$ (terms that are proportional to K^2), $\overline{f_{hom}^{(4)}}$ (terms involving coefficients of Eq.(3.14)), and $\overline{f_{in}^{(4)}}$ (terms involving coefficients of Eq. (3.15)); to obtain $2\tilde{\beta}+\tilde{\beta}_3$, we can use the relation

$$2\tilde{\beta}+\tilde{\beta}_3 = \overline{f_{hom}^{(4)}} + \overline{f_{in}^{(4)}} - \frac{\overline{h_s^2}}{8\pi}. \quad \text{(B7)}$$

The following is the full listing of $\overline{h_s^2}/8\pi$, $\overline{f_{hom}^{(4)}}$, and $\overline{f_{in}^{(4)}}$ (note that we have set β_3 of Eq. (3.14) to be zero):

$$\begin{aligned}
\frac{\overline{h_s^2}}{8\pi} &= \frac{8\pi^3 K^2}{\Phi_0^2}[(1-3\delta/\sqrt{2}+2\delta^2)^2\langle|\phi_0|^4\rangle \\
&+ 2(2\delta^2-\delta/\sqrt{2})(1-3\delta/\sqrt{2}+2\delta^2)\langle|\phi_0|^2|\phi_1|^2\rangle \\
&+ 2\delta^2(1-3\delta/\sqrt{2}+2\delta^2)\langle|\phi_0|^2|\phi_2|^2\rangle \\
&+ (2\delta^2-\delta/\sqrt{2})^2\langle|\phi_1|^4\rangle \\
&+ 2\delta^2(2\delta^2-\delta/\sqrt{2})\langle|\phi_1|^2|\phi_2|^2\rangle + \delta^4\langle|\phi_2|^4\rangle \\
&+ (1-3\delta/\sqrt{2}+2\delta^2)^2\langle|\phi_0|^2|\tilde{\phi}_0|^2\rangle \\
&+ 2(2\delta^2-\delta/\sqrt{2})(1-3\delta/\sqrt{2}+2\delta^2)\langle|\phi_0|^2|\tilde{\phi}_1|^2\rangle \\
&+ 2\delta^2(1-3\delta/\sqrt{2}+2\delta^2)\langle|\phi_0|^2|\tilde{\phi}_2|^2\rangle \\
&+ (2\delta^2-\delta/\sqrt{2})^2\langle|\phi_1|^2|\tilde{\phi}_1|^2\rangle \\
&+ 2\delta^2(2\delta^2-\delta/\sqrt{2})\langle|\phi_1|^2|\tilde{\phi}_2|^2\rangle \\
&+ \delta^4\langle\langle|\phi_2|^2|\tilde{\phi}_2|^2\rangle], \quad \text{(B8)} \\
\overline{f_{hom}^{(4)}} &= \beta_1(\langle|\phi_0|^4\rangle + \delta^4\langle|\phi_2|^4\rangle) + 2\beta_1'\delta^2\langle|\phi_0|^2\rangle\langle|\phi_2|^2\rangle \\
&- \beta_2(\langle|\phi_0|^2|\tilde{\phi}_0|^2 + \delta^4\langle|\phi_2|^2|\tilde{\phi}_2|^2\rangle) \\
&- 2\delta^2\beta_2'\langle|\phi_0|^2|\tilde{\phi}_2|^2\rangle, \quad \text{(B9)}
\end{aligned}$$

and

$$\begin{aligned}
\overline{f_{in}^{(4)}} &= \frac{2\gamma}{l^2}(\langle|\phi_0|^2|\tilde{\phi}_0|^2\rangle - \langle|\phi_0|^2|\tilde{\phi}_1|^2\rangle \\
&+ 3\delta^2\langle|\phi_0|^2|\tilde{\phi}_2|^2\rangle - 3\delta^2\langle|\phi_0|^2|\tilde{\phi}_3|^2\rangle) \\
&+ \frac{2\gamma'}{l^2}(3\delta^2\langle|\phi_0|^2|\tilde{\phi}_2|^2\rangle - 3\delta^2\langle|\phi_0|^2|\tilde{\phi}_3|^2\rangle \\
&+ 2\delta^4\langle|\phi_0|^2|\tilde{\phi}_1|^2\rangle + \delta^4\langle|\phi_0|^2|\tilde{\phi}_2|^2\rangle - 3\delta^4\langle|\phi_0|^2|\tilde{\phi}_3|^2\rangle \\
&+ 2\delta^4\langle|\phi_1|^2|\tilde{\phi}_1|^2\rangle + \delta^4\langle|\phi_1|^2|\tilde{\phi}_2|^2\rangle - 3\delta^4\langle|\phi_1|^2|\tilde{\phi}_3|^2\rangle \\
&+ 3\delta^4\langle|\phi_2|^2|\tilde{\phi}_2|^2\rangle - 3\delta^4\langle|\phi_2|^2|\tilde{\phi}_3|^2\rangle). \quad \text{(B10)}
\end{aligned}$$

APPENDIX C: CORRELATION FUNCTIONS

In calculating $\langle f_{in}^{(4)}\rangle$, note

$$(\boldsymbol{D}f)\cdot(\boldsymbol{D}g)^* = \frac{1}{l^2}[(\Pi_+ f)(\Pi_+ g)^* + (\Pi_- f)(\Pi_- g)^*] \quad \text{(C1)}$$

and

$$\begin{aligned}
\Pi_+\phi_n &= \sqrt{n+1}\phi_{n+1} \\
\Pi_-\phi_n &= \sqrt{n}\phi_{n-1}. \quad \text{(C2)}
\end{aligned}$$

Together with partial integration

$$\begin{aligned}
\langle(\Pi_+\phi_n)\tilde{\phi}_m^*\tilde{\phi}_p\phi_q^*\rangle &= \langle\phi_n(\Pi_-\tilde{\phi}_m)^*\tilde{\phi}_p\phi_q^*\rangle \\
&- \langle\phi_n\tilde{\phi}_m^*(\Pi_+\tilde{\phi}_p)\phi_q^*\rangle \\
&+ \langle\phi_n\tilde{\phi}_m^*\tilde{\phi}_p(\Pi_-\phi_q)^*\rangle, \quad \text{(C3)}
\end{aligned}$$

these equation gives

$$\begin{aligned}
\frac{1}{l^2}\langle\phi_0^*\tilde{\phi}_0(\boldsymbol{D}\phi_0)\cdot(\boldsymbol{D}\tilde{\phi}_0)^*\rangle &= \langle|\phi_0|^2|\tilde{\phi}_0|^2\rangle - \langle|\phi_0|^2|\tilde{\phi}_1|^2\rangle \\
\frac{1}{l^2}\langle\phi_0^*\tilde{\phi}_2(\boldsymbol{D}\phi_0)\cdot(\boldsymbol{D}\tilde{\phi}_2)^*\rangle &= \langle|\phi_0|^2|\tilde{\phi}_2|^2\rangle - \langle|\phi_0|^2|\tilde{\phi}_3|^2\rangle \\
\frac{1}{l^2}\langle\phi_2^*\tilde{\phi}_2(\boldsymbol{D}\phi_2)\cdot(\boldsymbol{D}\tilde{\phi}_2)^*\rangle &= 2\langle|\phi_0|^2|\tilde{\phi}_1|^2\rangle + \langle|\phi_0|^2|\tilde{\phi}_2|^2\rangle \\
&+ 2\langle|\phi_1|^2|\tilde{\phi}_1|^2\rangle + \langle|\phi_1|^2|\tilde{\phi}_2|^2\rangle \\
&- 3\langle|\phi_0|^2|\tilde{\phi}_3|^2\rangle - 3\langle|\phi_1|^2|\tilde{\phi}_3|^2\rangle \\
&+ 3\langle|\phi_2|^2|\tilde{\phi}_2|^2\rangle - 3\langle|\phi_2|^2|\tilde{\phi}_3|^2\rangle. \quad \text{(C4)}
\end{aligned}$$

These can be evaluated using

$$\begin{aligned}
\frac{\langle|\phi_p|^2|\phi_q|^2\rangle}{\langle|\phi_0|^2\rangle^2} &= \sum_{r,s} L_p^0(\boldsymbol{k}_{rs}^2/2)L_q^0(\boldsymbol{k}_{rs}^2/2)e^{-\boldsymbol{k}_{rs}^2/2}, \\
\frac{\langle|\phi_p|^2|\tilde{\phi}_q|^2\rangle}{\langle|\phi_0|^2\rangle^2} &= \sum_{r,s} L_p^0(\boldsymbol{k}_{rs}^2/2)L_q^0(\boldsymbol{k}_{rs}^2/2)e^{-\boldsymbol{k}_{rs}^2/2}\cos(\boldsymbol{k}_{rs}\cdot\boldsymbol{\tau}) \quad \text{(C5)}
\end{aligned}$$

where L_n^0 is a Laguerre polynomial of nth order and $\boldsymbol{k}_{rs} = (\sqrt{2\pi\sigma}r, \sqrt{2\pi/\sigma}(s-\varsigma r))$

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