Lect 3. Topological QFT.

I. The context.

- Caveats of $d \rightarrow d+\frac{1}{2}$ mapping

  $d \rightarrow d+\frac{1}{2}$

  - the "d" side (classical field theory)
    
    GL free energy functional (coarse grained effective Hamiltonian)

    \[ \ln Z = \beta \mathcal{H}[\psi] = \int d^d x \frac{k}{2} \left| \nabla \psi \right|^2 + \frac{\epsilon}{2} \left| \psi \right|^2 + \bar{\psi} \left( \mathbb{1} + m \right) \psi \]

    No dynamics

    - the "$d+\frac{1}{2}$" side

    \[ \left[ \tilde{\psi}(x), \tilde{\psi}^*(x') \right] = i \delta(x-x') \]

      conjugate wave function \( \bar{\psi} \), i.e. \( \tilde{\psi} \)

    \[ \mathcal{S}[\psi, \bar{\psi}] = \int \mathcal{D} \tilde{\psi} \mathcal{D} \psi \left[ e^{\int \bar{\psi} \left( \mathbb{1} + m \right) \psi - \frac{k}{2} \left| \nabla \psi \right|^2 - \frac{\epsilon}{2} \left| \psi \right|^2} \right] \]

    like \( P \bar{\psi} \), is imaginary

  \( d \rightarrow d+\frac{1}{2} \) for the thermodynamic properties of the QFT

  \( \Leftarrow \) treat the R.H.S of Eq (9) like Eq (1) in higher dim.
Caveats. * Mapping holds for thermodynamics only
  * Resulting classical system can be unusual and anisotropic (2.41)
  * Extra complication w/ no classical counterpart may arise
    e.g. Berry phases, sign problem
  * There are QPT's not involving O.P.
    e.g. Level crossing without symmetry change

Topo order

- Conventional order
  - Degeneracy in the GS Hilbert space.
    1) Degenerate states are related by local symmetry operation
      \[ \uparrow \uparrow \uparrow \uparrow \Rightarrow \downarrow \downarrow \downarrow \downarrow \]
  - Can be detected through local O.P.? 0
- Framework for studying QPT?
An exactly solvable model

1) (infinitely) many conserved quantities implies operators that commute with \( \hat{H} \)

2) Could be artificial

3) The knowledge of the entire spectrum is precious

  more than a Hamiltonian reverse engineered from
  a desired ground state wave function
  
  \[ \text{i.e. } \exists \text{ a Hamiltonian } \hat{H} \text{ such that } \hat{H} |\psi\rangle = 0 \]

  can open doors to finite T phase diagram

4) Can gain some insight into how to think about cases belonging to
   the same "universality class."

\[
\Rightarrow \text{Reasons for working out with the transverse field Ising model when}\n\text{first learning conventional QFT.}
\]
Topological QPT in the CSL model (in prep, S.-B. Chung, H. Yao, T. L. Hughes, EAK)

The Model

  \[ H = \sum_{\alpha=\{x,y,z\}} \sum_{\langle i,j \rangle_{\alpha\text{-link}}} J_{\alpha} \hat{S}^\alpha_i \hat{S}^\alpha_j - \sum_{\alpha=\{x,y,z\}} h_{\alpha} \hat{S}^\alpha_i \]
  - A "spin model" with very anisotropic exchange interaction
  - Exactly solvable for \( h=0 \).
  - Gapped phase turn non-Abelian for \( h \to 0 \) (perturbative)

- CSL model (Yao-Kivelson PRL 99, 247203 (2007))
  - Decorate vertices with triangles
  \[ H = \sum_{\alpha=\{x,y,z\}} \left( \sum_{\text{\alpha-links}} J_{\alpha} \hat{S}^\alpha_i \hat{S}^\alpha_j \right) + \sum_{\gamma=\{x',y',z'\}} \left( \sum_{\text{\gamma-links}} J_{\gamma} \hat{S}^\gamma_i \hat{S}^\gamma_j \right) \]
  - GS always break TRS
  - Define \( \phi_{\alpha} \equiv \hat{S}^x_0 \hat{S}^y_0 \hat{S}^z_0 \)
  \[ [\hat{H}, \hat{\phi}_{\alpha}] = [\hat{\phi}_{\alpha}, \hat{\phi}_{\beta}] = 0 \]
  \( \phi_{\beta} \)'s are conserved quantities
  \( \phi_{\alpha} \) is a triangle
Solving CSL model using Majorana fermion rep. of quantum spins.

- Majorana fermion rep. \( \delta_j^x = \chi_j d_j^x \)

where \( \{ c_j^x, c_j^y \} = 2 \delta_j^x, \delta_j^y - \delta_j^x \delta_j^y \) (\( \delta_j^x, \delta_j^y \) is \( 2 \delta_j \chi_j \chi_j^\dagger \))

\( \delta_j^x = c_j^x , \delta_j^y = d_j^x \)

under constraint

\[ \delta_j^x \delta_j^y \delta_j^z c_j = -1 \quad \Rightarrow \quad \delta_j^x \delta_j^y \delta_j^z = i \quad \text{for spin } 1/2 \]

\[ H \left[ \{ \hat{u}_{ij} \} \right] = J \sum_{\text{link}} \hat{u}_{ij} \chi_j c_i + \sum_{\text{x-link}} \hat{u}_{ij} \chi_i c_j \]

\( \hat{u}_{ij} = -\chi_i d_j^x \) for the given link, \( \left[ \hat{u}_{ij}, \hat{u}_{kl} \right] = 0, \left[ \hat{u}_{ij}, \hat{u}_{kl} \right] = 0 \)

takes \( \pm 1 \) values. \( \Rightarrow \) replace \( \hat{u}_{ij} \)'s with \( \hat{u}_{ij} = \pm 1 \)

A quadratic Hamiltonian

- Diagonalize \( H[\{ u_{ij} \}] = \sum_{n, \bar{n}} e_n \bar{n} [\{ u_{ij} \}; g] (b_n^\dagger b_n - 1/2) \), (3)

- Turns out vortices always (for all \( g \)) cost finite energy

- Focus on uniform flux state \( \Delta \) or \( \circ \) at low energy. Vortex-free spectrum has a gap \( \Delta(g) \)

\[ \Delta(g) \geq 0 \]

\[ v = \pm 1 \quad v = 0 \]

\[ g = J/J \]
A change in the topological degeneracy at the top QPT.

- Count only the states that are physical for the quantum spin model
  \[ d^2 \hat{d}^y d^2 \hat{c}^x = -1 \]
  \[ \Rightarrow \text{a constraint for Majorana fermions} \]

- Use a projector to annihilate unphysical states

\[ \hat{P} = \prod \left( 1 - \frac{d^2 \hat{d}^y d^2 \hat{c}^x}{2} \right) \]

- The effect of projector can depend on model parameters

  \[ \Rightarrow \text{can change the physical part of the spectrum} \]

- pre-projection: Four inequivalent global flux states are degenerate

  post-projection:

<table>
<thead>
<tr>
<th>( g )</th>
<th>( g &lt; g_c )</th>
<th>( g &gt; g_c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>vortex statistics</td>
<td>Non-Abelian</td>
<td>Abelian (Boson/fermion)</td>
</tr>
<tr>
<td>( g ) states on atoms</td>
<td>3</td>
<td>4</td>
</tr>
</tbody>
</table>

\[ g = \frac{j}{j'} \]

[Diagram showing spectrum change]
- Weak (BCS) to Strong (BEC) equal spin paired Pfip superfluid (at the many-field BDEP Hamiltonian level)
- Moore-Read state for $\nu = \frac{5}{2}$ Quantum Hall state on the non-Abelian side
- Honeycomb lattice Kitaev model under external field
- toric code on the Abelian side

<table>
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<tr>
<th>Vortex statistics</th>
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<td>G. state deg on atom</td>
<td>$3$</td>
<td>$4$</td>
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<tr>
<td>CSL model</td>
<td>$\mu &lt; \mu_c$</td>
<td>$\mu &gt; \mu_c$</td>
</tr>
<tr>
<td>Esp Pfip SF (BDEP)</td>
<td>$\nu \neq \nu_c$ (BCS)</td>
<td>$\mu &gt; \mu_c$ (BEC)</td>
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<tr>
<td>$\nu = \frac{5}{2}$ QH</td>
<td>Moore-Read</td>
<td></td>
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<tr>
<td>Kitaev model $h &gt; 0$</td>
<td>$h &gt;$</td>
<td>$h &lt;$</td>
</tr>
<tr>
<td>Z2 Matter + gauge</td>
<td></td>
<td>Kitaev toric code</td>
</tr>
</tbody>
</table>
Threading global flux.

- A global flux can be threaded through a large loop $\Gamma$.
  1) Create a pair of vortices
  2) transport along $\Gamma'$ by flipping bonds $\perp \Gamma'$
  3) annihilate the pair

Leaves the system in the uniform flux state

$\implies$ no energy cost $\rightarrow$ Degenerate Ground state.

- Four inequivalent global flux states

$(\Phi_x, \Phi_y) = (1,1), (1,-1), (-1,1), (-1,-1)$
- Global flux expectation value

- Finding physical states for each \((\bar{E}_x, \bar{E}_y)\) config
  1. Diagonalize \(\hat{H}[\{U_{ij}\}]\) for the given \(\{U_{ij}\}\)
     - Requires finding the complex fermion eigenstates by diagonalizing a 6x6 matrix \(\hat{H}([\psi], \{U_{ij}\})\)
     - Obtain the spectrum \(E_n([\psi])\), \(n=1,2,3\) for complex fermions \(\psi\)
  2. Many body eigenstates for given \(\{U_{ij}\}\)
     - Occupy or leave out each single particle states
  3. Hit the many body eigenstates with the projector
     \[ \hat{P}|n\rangle = \begin{cases} 0 & \text{unphysical} \\ |n\rangle & \text{physical} \end{cases} \]

- Ground state degeneracy can be affected by projection

\[ \hat{P} \text{ kills states with definite fermion parity, even/odd depends on } (\bar{E}_x, \bar{E}_y) \]

because

\[ 2^N \hat{P} = 1 + \prod_i (-d_i^+ d_i^z c_i) \left[ 1 + \sum_j (-d_j^+ d_j^z c_j) + \cdots \right] \]

\[ \propto \left[ 1 + \prod_{\text{even}} (-d_i^+ d_i^z c_i) \prod_{\text{odd}} \hat{U}_{mn} \right], \quad (4) \]

using \((d_i^+ d_i^y d_i^z c_i)^2 = 1\)
- With the change of spectrum with tuning $g$ through $g_c$, if $\hat{P}$ kills odd states for all $(\Phi_x, \Phi_y)$ for $g > g_c$ and odd states for $(1, 1), (1, -1), (-1, 1)$ even states for $(1, -1)$ for $g < g_c$.

- Define global flux expectation value

$$\langle \Phi_\alpha(T) \rangle \equiv \frac{1}{\mathcal{Z}} \text{tr} \Phi_\alpha e^{-\mathcal{H}/T}. \quad (6)$$

Clearly at $T=0$,

$$\langle \Phi_x(T=0) \rangle = \begin{cases} 
1/3 & (nA, g < g_c) \\
0 & (A, g > g_c)
\end{cases}. \quad (8)$$

Notice the expectation value is related to the topological degeneracy

$$n_{\text{DEG}} = 4 - 3\langle \Phi_x(T=0) \rangle. \quad (9)$$
Finite temperature crossover.

Considering that vortex gap remains finite throughout, focus on uniform flux sectors near critical point.

\[
\langle \Phi_\alpha \rangle = \frac{\sum_{\Phi_x, \Phi_y = \pm 1} \Phi_\alpha \mathcal{Z}(\Phi_x, \Phi_y)}{\mathcal{Z}},
\]

We can define \( T^\ast \) through the exponential decay of \( \Phi_x(T) \)
- \( T^* \) turns on upon \( g < g_c \) (Non-Abelian side).

![Graph with a phase transition]

- Gap in the uniform flux state spectrum.
  - Shows both side of topo QPT has finite gap.
  - Both has topo degeneracy.

\[ \Rightarrow \text{This is a QPT between two distinct topo phases} \]

- Finite size dependence

\[ T^* \sim \frac{\Delta(g)}{\ln N}, \quad (10) \]

- Decays in thermodynamic limit, but slower than any other quantity in the problem.

Why \( \sim \frac{1}{\ln N} \)?

- You can show \( \langle \text{Ex} \rangle \approx \text{const.} \Delta(g) / 2T \)
  - In the limit, \( T \ll A, N \) large, show \( T^* \sim \frac{1}{\ln N} \)
Open Questions

- Do we expect any power law in $T^*(g)$?
- Do these observations bear implications beyond this particular case we studied?
- Though in this case we know why $T^* < \frac{1}{16\pi}$, is this something to be expected in general in 2D top order?

The manuscript is under preparation by S.-B. Chung, H. Yao, T. Hughes, EAK.

① If you want to know more detail, stay tuned to arXiv.
② If you have good idea about above questions, email me!!
or write a paper on it. (1)