Coherent Superconductivity with a Large Gap Ratio from Incoherent Metals

Aavishkar A. Patel,[1,2] Michael J. Lawler,[3,4] and Eun-Ah Kim[3]
[1]*Department of Physics, Harvard University, Cambridge, Massachusetts 02138, USA*
[2]*Kavli Institute for Theoretical Physics, University of California, Santa Barbara, California 93106-4030, USA*
[3]*Department of Physics, Cornell University, Ithaca, New York 14853, USA*
[4]*Department of Physics, Binghamton University, Vestal, New York 13850, USA*

(Received 11 June 2018; published 30 October 2018)

A mysterious incoherent metallic (IM) normal state with T-linear resistivity is ubiquitous among strongly correlated superconductors. Recent progress with microscopic models exhibiting IM transport has presented the opportunity for us to study new models that exhibit direct transitions into a superconducting state out of IM states within the framework of connected Sachdev-Ye-Kitaev "quantum dots." Here, local Sachdev-Ye-Kitaev interactions within a dot produce IM transport in the normal state, while local attractive interactions drive superconductivity. Through explicit calculations, we find two features of superconductivity arising from an IM normal state. First, despite the absence of quasiparticles in the normal state, the superconducting state still exhibits coherent superfluid transport. Second, the nonquasiparticle nature of the IM Green's functions produces a large enhancement in the ratio of the zero-temperature superconducting gap Δ and transition temperature T_{SC}, $2\Delta/T_{\mathrm{SC}}$, with respect to its BCS value of 3.53.

DOI: 10.1103/PhysRevLett.121.187001

Superconductivity in correlated systems often emerges from a mysterious incoherent metallic (IM) state with T-linear resistivity. The origin of the T-linear resistivity has been a subject of active research and debate [1–4]. Moreover, Refs. [5–7] have pointed out that superconductivity emerging out of such strange metals should be qualitatively different from that emerging out of conventional metals. Nevertheless, the lack of a solvable microscopic model has prevented the community from forming a concrete connection between many inexplicable properties of the superconducting state and the IM state in correlated systems.

Recent proposals of microscopic models exhibiting IM transport in a solvable limit [8–14] present new avenues. The approach shared among these models is to build on Sachdev and Ye [15] finding non-Fermi liquid Green's functions in a solvable model of fermions with infinite-range interactions. Although both this original model and a simpler model with Majorana fermions [16] exhibit non-Fermi liquid properties as well as interesting connections to quantum gravity [16,17] in the solvable limit, they do not support local current operators. However, by introducing local coupling between multiple copies of these infinite-range models, in the spirit of weakly coupled quantum dots each hosting multiple orbitals, Refs. [8–13] established solvable microscopic models with IM transport. These models led to new insights regarding the loss of quasiparticle coherence during scattering leading to such transport. But moreover, they have put us in an opportune moment to theoretically study the properties of superconducting (SC) phases born out of such IMs, in a solvable limit.

In this Letter, we consider two models that can be solved in a large-N limit that demonstrate the much sought after transition from an IM with T-linear resistivity to SC. We then study the implication of strong correlations destroying coherent quasiparticles on the superconducting transition and state. In spite of the incoherent normal state, the paired state still supports a coherent supercurrent. We further show that a key prediction of the Bardeen-Cooper-Schrieffer (BCS) mean-field theory [18] of superconductivity is violated: the ratio between the zero-temperature gap Δ and the transition temperature T_{SC} far exceeds the BCS value of $2\Delta/T_{\mathrm{SC}} \approx 3.53$. We compare this mechanism of gap ratio enhancement with that in the Eliahsberg theory and in experiments.

Model 1.—We consider a lattice model of two species of fermions a, b with disordered local on-site Sachdev-Ye-Kitaev interactions of 4th order (SYK$_4$), but with a *uniform* quadratic hopping, and an attractive term that pairs the two species locally (Fig. 1). It is given by

$$\begin{aligned} H_1 = &\sum_m \sum_{i_1,\ldots,i_4=1}^{N} [K^{am}_{i_1,\ldots,i_4} a^\dagger_{i_1 m} a^\dagger_{i_2 m} a_{i_3 m} a_{i_4 m} + (a \leftrightarrow b)] \\ &- t \sum_{\langle mn \rangle} \sum_{i=1}^{N} [a^\dagger_{im} a_{in} + (a \leftrightarrow b) + \mathrm{H.c.}] \\ &- \frac{U}{N} \sum_m \sum_{i,j=1}^{N} b^\dagger_{im} a^\dagger_{im} a_{jm} b_{jm}, \end{aligned} \tag{1}$$

where m and n are the site indices with N fermions of each type. Here, the disordered complex Gaussian random

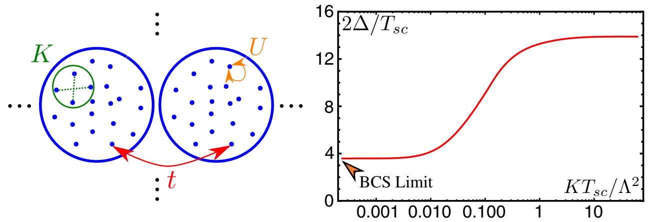

FIG. 1. (Left) A cartoon of model 1. (Right) A plot of $2\Delta/T_{\mathrm{SC}}$ as $T \to 0$, vs $KT_{\mathrm{SC}}/\Lambda^2$ for $K = 1000$, $T_{\mathrm{SC}} = 10$, for different values of the bandwidth $\Lambda \sim t$ in model 1. The value of U is adjusted as Λ is varied in order to keep T_{SC} fixed. For large Λ, the transition to SC is from a dispersive Fermi liquid, and we find the BCS result $2\Delta \approx 3.53T_{\mathrm{SC}}$. For small values of Λ, such that $KT_{\mathrm{SC}}/\Lambda^2 \gg 1$, the transition to SC is from a non-Fermi liquid IM with a linear-in-T resistivity, and $2\Delta \gg 3.53T_{\mathrm{SC}}$.

couplings $K^{\alpha m}_{i_1,\ldots,i_4}$ satisfy $\langle\langle K^{\alpha m}_{i_1,i_2,i_3,i_4} K^{\beta n}_{i_4,i_3,i_2,i_1} \rangle\rangle = K^2/(8N^3)\delta_{\alpha\beta}\delta_{mn}$, where $\langle\langle \cdots \rangle\rangle$ denotes disorder averaging, and all other averages are zero. A simpler model, without the attractive U term, and with only one type of fermion, was first proposed in Ref. [11].

Without the U term, and with $K \gg t$, this model exhibits a crossover between a high temperature IM state with T-linear resistivity to a low temperature Fermi liquid state. Specifically, for $K \gg T \gg t^2/K$, the Green function is asymptotically given by the local SYK Green's function in imaginary time [17]:

$$G^{a,b}(0 < \tau < \beta) = -\frac{\pi^{1/4}}{K^{1/2}\sqrt{2}} \left(\frac{T}{\sin(\pi T \tau)} \right)^{1/2}. \quad (2)$$

Using the Kubo formula, one can derive the linear-in-T resistivity in the large-N limit from the scaling of the above Green's function [10,12,13]. On the other hand, for $T \ll t^2/K$, the Green function approaches that of a Fermi liquid whose resistivity scales as T^2 [11,13].

The attractive U term leads to a spatially uniform s-wave pairing instability at $T = T_{\mathrm{SC}}$. Once SC is established, the order parameter $\Delta_0 = \langle \sum_i a_{im} b_{im} \rangle / N$ condenses. In the large-N limit, we then get the Dyson and gap equations (Supplemental Material [19])

$$\begin{aligned} \mathcal{G}(i\omega_n) &= \int \frac{d^d k}{(2\pi)^d} \frac{G(i\omega_n, k)}{1 + U^2 |\Delta_0|^2 |G(i\omega_n, k)|^2}, \\ \Sigma(\tau - \tau') &= -K^2 \mathcal{G}^2(\tau - \tau') \mathcal{G}(\tau' - \tau), \\ G^{-1}(i\omega_n) &= i\omega_n - \xi_k - \Sigma(i\omega_n), \\ T \sum_{\omega_n} \int \frac{d^d k}{(2\pi)^d} & \frac{|G(i\omega_n, k)|^2 \Delta_0}{1 + U^2 |\Delta_0|^2 |G(i\omega_n, k)|^2} = \frac{\Delta_0}{U}, \end{aligned} \quad (3)$$

which can be iterated numerically starting with an infinitesimal Δ_0 and the free fermion $G(i\omega_n) = (i\omega_n)^{-1}$ in order to determine both G and Δ_0. Here, $\mathcal{G}(\tau) = \mathcal{T}_\tau \langle \sum_i a_{im}(\tau) a^\dagger_{im}(0) \rangle / N = \mathcal{T}_\tau \langle \sum_i b_{im}(\tau) b^\dagger_{im}(0) \rangle / N$ is the local time-ordered Green's function, and ξ_k is the dispersion of the fermions.

For simplicity, while still capturing the essential physics, we consider $d = 2$, and $\xi_k = \Lambda k^2/(4\pi) - \Lambda/2 \equiv \epsilon_k - \Lambda/2$, with $\epsilon_k \in [0, \Lambda]$, where $\Lambda \sim t$ is the bandwidth of the dispersion. We can then replace $\int [d^2 k/(2\pi)^2] \to (1/\Lambda) \int_{-\Lambda/2}^{\Lambda/2} d\xi_k$, and perform all momentum integrations analytically. In general, we determine T_{SC} numerically by taking the limit $\Delta_0 \to 0$ in the last line of (3). For $T > T_{\mathrm{SC}}$, the solution of (3) with $\Delta_0 = 0$ corresponds to a stable local minimum of the free energy. The large-N limit strongly suppresses fluctuations of Δ_0 out of this minimum. As T is lowered below T_{SC}, the curvature of the free energy as a function of Δ_0 at $\Delta_0 = 0$ changes sign, and the system condenses to a new minimum with $\Delta_0 \neq 0$ (Supplemental Material [19]).

When U is infinitesimal so that $T_{\mathrm{SC}} \ll t^2/K$, the SC arises out of a Fermi liquid, and we find the standard BCS result of $2\Delta = 3.53T_{\mathrm{SC}}$. On the other hand if both K and U are large such that $K \gg T_{\mathrm{SC}} \gg t^2/K$, we obtain the transition to SC from the linear-in-T IM. From (3) and (2) we get (Supplemental Material [19])

$$T_{\mathrm{SC}} \approx \frac{2K}{\pi} \tan^{-1}(e^{-\pi^{1/2} K/U}), \quad (4)$$

where we employed a UV frequency cutoff $\sim K$ in (3). By solving (3) numerically, we can study how the gap ratio evolves through the crossover between SC emerging from a Fermi liquid to SC emerging from a linear-in-T IM. For this, we study the variation of the zero-temperature gap to single-particle excitations Δ, with the bandwidth Λ, keeping K and T_{SC} fixed. Δ corresponds to the location of the peak of the spectral function $\mathcal{A}(\omega, \{k : \xi_k = 0\})$, with

$$\mathcal{A}(\omega, k) = -2\mathrm{Im}\left(\frac{G_R(\omega, k)}{1 + U^2 |\Delta_0|^2 G_R(\omega, k) G_R^*(-\omega, k)} \right), \quad (5)$$

which may be obtained from a numerical solution of the real-time version of (3) (Supplemental Material [19]). For small interactions and T_{SC} (relative to the bandwidth), SC emerges from a Fermi liquid, and the gap ratio is consistent with the "BCS" value of $2\Delta \approx 3.53T_{\mathrm{SC}}$. However, the interactions U and K are cranked up relative to the bandwidth so that SC emerges directly from a T-linear IM, the gap ratio substantially exceeds the BCS gap ratio (Fig. 1). When the gap ratio is enhanced significantly, we also find that the superconducting transition becomes first order; i.e., the order parameter Δ_0 jumps discontinuously to a nonzero value at $T = T_{\mathrm{SC}}$. First-order transitions have also been noted earlier in studies of superconductivity arising from non-Fermi liquids [20,21].

Model 2.—We now consider a model that realizes an instability to SC from a non-Fermi liquid even for infinitesimal values of U. In order to avoid a Fermi liquid state as

$T \to 0$, we need a model that has no quadratic terms in its Hamiltonian. We, however, still want a linear-in-T resistivity above some small temperature scale. We thus replace the on-site interactions of model 1 with higher order SYK_8 terms, and the quadratic hopping between adjacent sites by pair hoppings that realize SYK_4 interactions between adjacent sites. As we shall explain in detail, the scaling dimensions of the current operator and the local Green's functions then lead to a linear-in-T resistivity above a certain temperature. Since the charge transfer between sites is now strongly disordered, we have to use an attractive interaction given by not a conventional on-site pairing term, but rather a spatially uniform term that simultaneously binds a-b pairs on site and hops them between nearest-neighbor sites, which allows coherent pair hopping below T_{SC} and hence establishes superfluid phase coherence in the SC state.

We start with a single-site SYK_8 model with two species of fermions:

$$H_{2,0} = \sum_{i_1,\ldots,i_8=1}^{N} (J^a_{i_1,\ldots,i_8} a^\dagger_{i_1} \cdots a^\dagger_{i_4} a_{i_5} \cdots a_{i_8} + J^b_{i_1,\ldots,i_8} b^\dagger_{i_1} \cdots b^\dagger_{i_4} b_{i_5} \ldots b_{i_8}), \tag{6}$$

with complex Gaussian random couplings $J^\alpha_{i_1,\ldots,i_8}$ satisfying $\langle\langle J^\alpha_{i_1,\ldots,i_8} J^{\alpha*}_{i_1,\ldots,i_8} \rangle\rangle = J^2/(2304N^7)$, with all other averages being zero. For $J \gg T$, the resulting SYK_8 Green's function is [9]

$$G^{a,b}(0<\tau<\beta) = -\frac{C_8}{J^{1/4}}\left(\frac{T}{\sin(\pi T\tau)}\right)^{1/4},$$
$$C_8 = \sin^{1/4}\left(\frac{\pi}{8}\right)\Gamma^{1/8}\left(\frac{1}{4}\right)\Gamma^{1/8}\left(\frac{7}{4}\right). \tag{7}$$

We then place a system described by (6) on each site of a lattice indexed by m. The random SYK_8 couplings are not correlated between sites. We introduce two intersite terms between nearest-neighbor sites: a random SYK_4 interaction that hops a and b fermions independently in pairs between nearest-neighbor sites which is necessary for IM transport, and a uniform hopping term for a-b Cooper pairs which drives superfluid phase coherence below T_{SC} [Fig. 2(a)]:

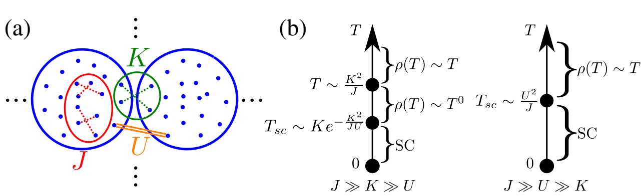

FIG. 2. (a) A schematic representation of model 2. (b) Crossover diagrams in different regimes. For $J \gg K \gg U$, there is first a crossover to an IM with T-independent resistivity, before the SC transition. For $J \gg U \gg K$, we just have a transition from an IM with T-linear resistivity to an SC.

$$H_2 = \sum_m \sum_{i_1,\ldots,i_8=1}^{N} [J^{am}_{i_1,\ldots,i_8} a^\dagger_{i_1 m} \cdots a^\dagger_{i_4 m} a_{i_5 m} \cdots a_{i_8 m} + (a \leftrightarrow b)]$$
$$+ \sum_{\langle mn\rangle} \sum_{i_1,\ldots,i_4=1}^{N} [K^{a\langle mn\rangle}_{i_1,\ldots,i_4} a^\dagger_{i_1 m} a^\dagger_{i_2 m} a_{i_3 n} a_{i_4 n} + (a \leftrightarrow b)$$
$$+ \mathrm{H.c.}] - \frac{U}{zN} \sum_{\langle m,n\rangle} \sum_{i,j=1}^{N} [b^\dagger_{im} a^\dagger_{im} a_{jn} b_{jn} + \mathrm{H.c.}], \tag{8}$$

with $\langle\langle K^{\alpha\langle m,n\rangle}_{i1,i2,i3,i4}, K^{\alpha\langle m,n\rangle*}_{i1,i2,i3,i4} \rangle\rangle = K^2/(8zN^3)$, and all other averages are zero. Note that the role of the SYK_4 intersite interactions in this model is distinct from the intrasite SYK_4 interactions in model 1. The coordination number of the regular lattice is z.

Normal state of model 2.—For $T > T_{\mathrm{SC}}$, due to the large-N limit, the Dyson equation for the fermion Green's functions is local in space and is simply given by

$$\Sigma(\tau) = -J^2 G^4(\tau) G^3(-\tau) - K^2 G^2(\tau) G(-\tau),$$
$$G^{-1}(i\omega_n) = i\omega_n - \Sigma(i\omega_n), \tag{9}$$

for both fermion types a and b. Defining the energy scaling dimensions $[a] = [b] = 1/4$ using the K terms in (8), we see that J is irrelevant at low energies, but dominates at high energies. This implies a crossover between two distinct IM states around $T \approx K^2/J$: An SYK_8 dominant regime with the Green's function given in Eq. (7) for $T \gg K^2/J$ and an SYK_4 dominant regime with the Green's function given in Eq. (2) for $T \ll K^2/J$. Depending on the strength of the attractive interaction U superconductivity will set in out of IM states with qualitatively different transport.

The current operator on the bond indexed by $\langle mn\rangle$, that leads to a conductivity *not* suppressed by $1/N$ is

$$I^{\widehat{mn}}_m = 2i \sum_{i1,\ldots,i4=1}^{N} [K^{a\langle mn\rangle}_{i1,\ldots i4} a^\dagger_{i1m} a^\dagger_{i2m} a_{i3n} a_{i4n} + (a \leftrightarrow b) - \mathrm{H.c.}] \tag{10}$$

(there is another contribution to the current from the U term, but it leads to a contribution to the conductivity that is not extensive in N in the IM). Using this, we then obtain the uniform, disorder averaged, current-current correlator at large N

$$\langle\langle \langle \hat{I} \hat{I} \rangle(\mathbf{q}=0,\tau) \rangle\rangle = 4\frac{NK^2}{z} G^2(\tau) G^2(-\tau). \tag{11}$$

For $T \gg K^2/J$, and we can approximate G to be the SYK_8 Green's function (7) to obtain

$$\sigma^J_{\mathrm{dc}} = 2C_8^4 \frac{NK^2}{zJT}. \tag{12}$$

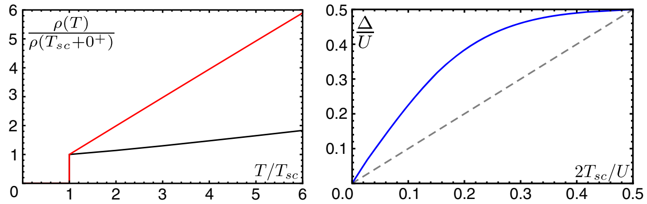

FIG. 3. (Left) Temperature dependence of the dc resistivity $[\rho(T)/\rho(T_{\rm SC}+0^+)]$, vs $T/T_{\rm SC}$ in model 2. (i) $J \gg K > U$, with a transition from a roughly T-independent resistivity to an SC (black), (ii) $J \gg U \gg K$, showing a direct transition from T-linear resistivity to an SC (red). The values of the parameters used are (i) $J = 100$, $K = 20$, $U = 13.8584$, and (ii) $J = 100$, $U = 9.0515$, $K = 1$. $T_{\rm SC} = 1$ in both cases. (Right) A plot of the normalized physical gap Δ/U as $T \to 0$ in model 2, obtained from the spectral function $\mathcal{A}(\omega)$ determined numerically (Supplemental Material [19]), vs the lower bound $\Delta_b = 2T_{\rm SC}/U$ (dashed line) for $J = 100$, $K = 1$, and different values of U.

Hence, this regime is an IM with T-linear resistivity. However, for $T \ll K^2/J$, the system will cross over to transport controlled by the SYK_4 Green's function (2) with $\sigma_{\rm dc}^K = N/z$ (i.e., a T-independent constant). Depending on the relative strength of the attractive interaction U in comparison to the SYK_4 interaction strength K, this T-independent resistivity IM may or may not be visible (see Fig. 3).

Superconductivity of model 2.—The attractive U term leads to a leading uniform ($\mathbf{q} = 0$) s-wave pairing instability in the IM phase, which can be seen by considering the renormalization of the U term of (8) in the pairing channel at different values of the external momentum $\mathbf{q}$, through the standard resummation of pairing bubbles. For infinitesimal U with $J \gg K \gg U$, we have a transition from an IM with an approximately T-independent resistivity to an SC. In this case the physics of SYK_4 controls $T_{\rm SC}$ and $T_{\rm SC}$ takes the same form as that in model 1, given by Eq. (4). A new case of interest is accessible when $J \gg U \gg K$. In this regime the superconducting transition occurs in the temperature range with T-linear resistivity and we obtain (Supplemental Material [19])

$$T_{\rm SC} = \frac{C_8^4 \Gamma^2(1/4)}{\pi \Gamma^2(3/4)} \frac{U^2}{J}. \tag{13}$$

At $T_{\rm SC}$, we then have a transition from an IM with a linear-in-T resistivity to an s-wave SC.

Now we can investigate implications of the IM normal state on the superconducting state. We find the correlation driven IM normal state affects the superconducting state through an enhancement of the gap ratio as in model 1. To see this without a BCS limit to benchmark, we consider the limit of vanishing SYK interactions, i.e., $U \gg J, K$. In this limit, the paired state becomes entropically unstable above transition temperature of $T_{\rm SC} = U/4$, and the normal state contains featureless free fermions. Further one can find analytically that the zero-temperature gap in this limit to be given by $\Delta_b = U/2 = 2T_{\rm SC}$. Now the implication of IM normal state is apparent in the numerically obtained value of Δ (see Fig. 3). Δ always exceeds the lower bound value of $\Delta_b = 2T_{\rm SC}$ (dashed line) in the presence of the J, K interactions. As in model 1, we find that the superconducting transition becomes first order when the J, K interactions are strong enough to enhance the gap ratio significantly.

Our models also show coherent superfluid transport despite incoherence driven by the SYK interactions in the normal state. In the SC phase of model 2, charge transport for $T \ll \Delta$ is controlled by gapless low energy phase fluctuations, whose Hamiltonian is derived by letting $\Delta_{0m} = \Delta_0 e^{i\theta_m}$,

$$H_\theta \approx \frac{NU}{z} |\Delta_0|^2 \sum_{\langle m,n \rangle} (\theta_m - \theta_n)^2 \to \frac{NU}{z} |\Delta_0|^2 \int_x (\nabla\theta)^2. \tag{14}$$

This implies the usual diamagnetic electromagnetic response at frequencies $|\omega| \ll \Delta$ and $T = 0$ [22], with

$$\lim_{\omega\to 0} \sigma_{ij}(\omega, \mathbf{q} = \mathbf{0}) \approx 4\delta_{ij} \frac{NU}{z} \frac{|\Delta_0|^2}{i\omega}, \tag{15}$$

and a superfluid phase stiffness that is extensive in N, uninhibited by incoherence in the normal state. A similar analysis confirms coherent superfluidity in the superconducting phase of model 1 (Supplemental Material [19]).

Conclusion.—We studied two models exhibiting superconducting transition out of an IM phase with T-linear resistivity within the framework of connected SYK "quantum dots." By having a solvable limit exhibiting this phenomena ubiquitous in correlated systems, we explicitly established implications of a strongly correlated incoherent normal state on superconductivity. The severe electron-electron scattering that destroys coherent quasiparticles and drives T-linear resistivity does not inhibit formation of a coherent superconducting state. Instead, the electron-electron scattering leads to dramatic enhancement in the gap ratio, while also driving the superconducting transition first order.

It is instructive to contrast the gap ratio enhancement seen in our IM-SC transition to that obtained in the standard Eliashberg theory of phonon-mediated superconductivity. Within the Eliashberg theory, a relatively gentle deviation of the measured gap ratio from the universal BCS value in elemental superconductors and alloys can be accounted for [23,24]. Such enhancement is driven by a suppression of the $T_{\rm SC}$ due to fluctuation effects ignored in the BCS mean-field theory. However, due to the large-N limit, the effects of retardation of the pairing interaction on the Fermion self-energy are suppressed, and the gap equations in (3) are actually exact. Thus, the enhancement of the gap ratio in our model occurs not due to the suppression of $T_{\rm SC}$ by the thermal fluctuations of the anomalous Green's function, but rather due to the quasiparticle nature of the IM Green's functions. It is worth noting that an enhancement of the gap ratio is also seen in holographic models of superconductors [25].

Interestingly, an extreme gap ratio enhancement is widely seen in various correlated-electron superconductors, such as in cuprates and iron-based superconductors [26,27]. Our work presents the first microscopic mechanism of such an enhancement that is not driven by the suppression of T_{SC} by pairing fluctuations, but rather through the redistribution of spectral weight of an incoherent, non-Fermi liquid normal state.

We acknowledge useful discussions with James Analytis, Leon Balents, Hong Ding, Steve Kivelson, and Subir Sachdev. This work was initiated during a KITP program supported by NSF Grant No. NSF PHY-1748958. A. A. P. was supported by the NSF Grant No. PHY-1125915 via a KITP Graduate Fellowship, and by NSF Grant No. DMR-1664842. E.-A. K. was supported by U.S. Department of Energy, Office of Basic Energy Sciences, Division of Materials Science and Engineering under Award No. DE-SC0010313.

[1] C. M. Varma, P. B. Littlewood, S. Schmitt-Rink, E. Abrahams, and A. E. Ruckenstein, Phys. Rev. Lett. **63**, 1996 (1989).
[2] S. A. Hartnoll, Nat. Phys. **11**, 54 (2015).
[3] S. A. Hartnoll and A. Karch, Phys. Rev. B **91**, 155126 (2015).
[4] J. A. N. Bruin, H. Sakai, R. S. Perry, and A. P. Mackenzie, Science **339**, 804 (2013).
[5] V. J. Emery and S. A. Kivelson, Nature (London) **374**, 434 (1995).
[6] V. J. Emery and S. A. Kivelson, Phys. Rev. Lett. **74**, 3253 (1995).
[7] J. Zaanen, Nature (London) **430**, 512 (2004).
[8] Y. Gu, X.-L. Qi, and D. Stanford, J. High Energy Phys. 05 (2017) 125.
[9] R. A. Davison, W. Fu, A. Georges, Y. Gu, K. Jensen, and S. Sachdev, Phys. Rev. B **95**, 155131 (2017).
[10] X.-Y. Song, C.-M. Jian, and L. Balents, Phys. Rev. Lett. **119**, 216601 (2017).
[11] P. Zhang, Phys. Rev. B **96**, 205138 (2017).
[12] A. A. Patel, J. McGreevy, D. P. Arovas, and S. Sachdev, Phys. Rev. X **8**, 021049 (2018).
[13] D. Chowdhury, Y. Werman, E. Berg, and T. Senthil, Phys. Rev. X **8**, 031024 (2018).
[14] X. Wu, X. Chen, C.-M. Jian, Y.-Z. You, and C. Xu, Phys. Rev. B **98**, 165117 (2018).
[15] S. Sachdev and J. Ye, Phys. Rev. Lett. **70**, 3339 (1993).
[16] A. Y. Kitaev, Entanglement in Strongly-Correlated Quantum Matter (2015), http://online.kitp.ucsb.edu/online/entangled15/kitaev/.
[17] S. Sachdev, Phys. Rev. X **5**, 041025 (2015).
[18] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. **108**, 1175 (1957).
[19] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.121.187001 for derivations of formulas.
[20] A. V. Chubukov, A. M. Finkel'stein, R. Haslinger, and D. K. Morr, Phys. Rev. Lett. **90**, 077002 (2003); **90**, 077002 (2003).
[21] I. Giannakis, D. Hou, H.-c. Ren, and D. H. Rischke, Phys. Rev. Lett. **93**, 232301 (2004).
[22] A. Altland and B. D. Simons, *Condensed Matter Field Theory* (Cambridge University Press, Cambridge, 2010).
[23] J. P. Carbotte, Rev. Mod. Phys. **62**, 1027 (1990).
[24] D. Dhokarh and A. V. Chubukov, Phys. Rev. B **83**, 064518 (2011).
[25] S. A. Hartnoll, C. P. Herzog, and G. T. Horowitz, Phys. Rev. Lett. **101**, 031601 (2008).
[26] M. Hashimoto, I. M. Vishik, R.-H. He, T. P. Devereaux, and Z.-X. Shen, Nat. Phys. **10**, 483 (2014).
[27] I. M. Hayes, R. D. McDonald, N. P. Breznay, T. Helm, P. J. W. Moll, M. Wartenbe, A. Shekhter, and J. G. Analytis, Nat. Phys. **12**, 916 (2016).